MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 93 No. 9. SUPPLEMENTARY NUMBER

SPECTROGRAPHIC STUDIES OF THE PLANETS.

(George Darwin Lecture, delivered by Dr. V. M. Slipher, Assoc.R.A.S., on 1933 May 12.)

Spectrum analysis may fairly be said to have entered upon its fruitful application to astronomy with the work of the great English pioneer in this field, Sir Wm. Huggins. In the early sixties of the last century he was inspired to take up this new study by Kirchhoff's interpretation of the dark lines of the solar spectrum, just then published. Up to that time it had not been thought possible that man could ever know the substances actually composing the distant heavenly bodies. But he, with rare vision, embraced this marvellous new means for their study and began analysing their light to read their secrets. And after seventy years of truly wonderful revelations this work is still going on with as fruitful results to-day as at any time in the past. The distance to the heavenly bodies, be it even millions of light years, is no obstacle to the definiteness of the message that may be carried by their spectrally analysed light.

Others also soon took up this new study of the stars, and quite naturally they must have eagerly sought to apply this new means of observation to the brighter planets.

It is just one hundred years ago that Sir David Brewster recognized absorption lines in the spectrum of sunlight that were due to the Earth's atmosphere. And what the air of the Earth did might be duplicated by the atmospheres of the other planets in accordance with their constitutions and conditions, and hence by their spectra they should show something as to their atmospheres. Sir Wm. Huggins and others in this branch of astronomy studied the spectra of the planets visually. And to Huggins we are indebted for the greater contribution of the first photographs of the blue and violet of the spectra of the planets. His efforts and those of others in this direction were confined to the more refrangible end of the spectrum, because plates

were not then efficient in the red and their observations there were of necessity made visually. In the short-wave end of the spectrum they found atmospheric absorptions, but in the long-wave portion, especially of the giant planets, there appeared selective absorptions that seemed peculiar to their atmospheres. The most striking of these was that in the spectra of Jupiter and Saturn near λ 6190, and this one and others in the yellow and green of the spectrum of Uranus, observed also by Keeler at the Lick Observatory. This brief reference to early work will perhaps suffice, as time is not available to give a proper review of the valuable studies of these early observers.

When I began my study of the spectra of the planets thirty years ago it was difficult to photograph the spectrum much beyond λ 6000 on the available commercial plates, and even by dyeing them with sensitizers it was for a time not possible to add much more than 500 more angstrom units of the red spectrum. Fortunately since then very great advances have been made in plates for photography in the red and infra-red. And to-day, thanks especially to the splendid work of Dr. C. E. K. Mees, Director of the Research Laboratory of the Eastman Kodak Company, and his friendly assistance, it is now possible to record the spectrum as far as to beyond λ 10,500 in the deep infra-red. The illustration showing the solar spectrum (Plate 15) will make clear the quite enormous gain that has been made in the range of wave-lengths that have been brought under observation since 1903. This gain is more important in the study of the planets than in any other field for the reason that their selective absorption rapidly increases into the longer wave-lengths, and the tendency still goes on growing through this latest addition of 1500 angstroms in the deep infra-red. So, then, we are still wishing to reach farther into the heat spectrum, with assurance that when such is possible further valuable knowledge will be added to what we know of planetary spectra. (Plate 14.)

During more than the last quarter of a century it has been my custom at the Lowell Observatory to re-photograph the spectra of the planets whenever it has been possible to reach a little farther into the longer wave-lengths, for every extension in that direction has added new planetary absorption bands. Research in this field has constituted a logical portion of the observatory's regular planetary programme, which, because of the observatory's exceptionally favourable site and suitable equipment, could be followed with expectation of fruitful results.

The light from all celestial objects must traverse the Earth's atmosphere in reaching our telescopes, and hence the spectrum of the distant planet will be crossed by the bands of oxygen and water-vapour. The more of our air the planet's light encounters, the more intense will be the resulting telluric absorptions. Thus, in order to reduce this Earth absorption as much as possible, it is clearly desirable to observe the planet when it is at its highest point in the sky and at a time when our air carries the least water-vapour. The elevation of the Lowell Observatory gives it an important advantage in that the densest and most absorptive 7250 feet of the Earth's atmospheric envelope are thereby surmounted. In more recent years our mountain station, elevation 11,400 feet, has given still further aid in this same direction.

DESCRIPTION OF PLATES 13–17

PLATE 13

The spectra of Jupiter, Saturn and Venus, with a powerful three-prism spectrograph, taken for rotation effects.

PLATE 14

FIG. 1.—The spectra of Mars and the Moon at identical altitudes, made when the Flagstaff air was excessively dry; hence the spectra show that the water-vapour band was stronger in Mars.

FIG. 2.—The central group of spectra gives a general view of the green, yellow and red of the four giant planets, compared with the spectrum of the Moon. These disclose the atmospheric absorption bands and show how they increase from Jupiter to Neptune. The plates used for Uranus and Neptune were not sensitive beyond λ 7200.

FIG. 3.—The spectrum of Jupiter is compared with that of the Sun in the red and infra-red. It will be noted that the wide and intense band which extends from λ 8800 to λ 9050 cuts off the spectrum of Jupiter very sharply, considerably short of the solar comparison spectra. Numerous other sharp, narrow and broad diffuse bands are in evidence.

PLATE 15

FIG. 1.—Normal spectrum (diagrammatic) showing new infra-red addition.

FIG. 2.—Two spectra of the Sun, the second one with the Sun at very low altitude, and hence it has very strong telluric absorptions. These spectra end in the infra-red near λ 10,500.

FIG. 3.—Shows the spectrum of Jupiter compared with that of the Sun, made with single-prism dispersion, and records the spectrum of the planet as far as λ 10,000. It will be noted that very strong absorption prevails in the deep infra-red and practically cuts off the spectrum of Jupiter considerably short of the solar comparison spectrum. With this low dispersion a general view of the atmospheric absorption of the planet is had. It will be noted that, in addition to the bands themselves, the spectrum of Jupiter is relatively much weaker with increase of wave-length from red to the infra-red, and suggests that Jupiter does not send us much light of longer wave-length than λ 10,000.

FIG. 4.—The spectrum of Saturn, made with reflecting telescope and low dispersion, to reveal the different reflectivity of the ball and rings of the planet between the ultra-violet and λ 7300. It is evident that the rings have relatively much higher reflectivity than the ball in the short wave-lengths, and (*vice versa*) the ball much higher reflectivity than the rings in the long wave-lengths.

FIG. 5.—The Earthshine spectrum. Very low dispersion spectra of light of the Moon's crescent, at the top, compared with the Earthshine spectrum below. It is seen that, whereas in the violet the spectra are comparable in intensity, the Earthshine spectrum in the longer wave-lengths is relatively much weaker than moonlight: this in spite of the fact that the light of the Earthshine has had its short wave-lengths more reduced in its additional travel in the Earth's atmosphere. This shows, what we already knew, that the Earth appears bluer than normal sunlight.

PLATE 16.—SPECTRAL RADIATIONS OF THE NIGHT SKY

FIG. 1.—Here are spectra, simultaneously photographed, of five different portions of the night sky, the middle one of the zenith sky, the other four of the sky at an altitude of about 12° from the four indicated points, east, south, north and west. This combined spectrogram, made 1932 May 29, shows the superposition of the spectrum of a brief auroral display that occurred on that night. The aurora naturally was, as the spectra show, very much stronger in the northern sky, although the stronger auroral radiations are traceable through the spectra all over the sky. This is a very unusual result for the spectrum of the night sky, for auroral spectral bands are very rare on the night-sky spectra.

FIG. 2.—This is what might be called the *normal spectrum* of the night sky, with the instrument adjusted as stated in fig. 1, and records simultaneously the spectra of the five indicated portions.

FIG. 3.—This displays the red portion of the spectrum of the night sky, made with two-prism dispersion, and shows partial resolution of the sky radiations into lines. In particular the forbidden oxygen lines at λ 6300 and λ 6363 are to be noted. Of longer wave-lengths, there are evident a considerable number of radiations for the resolution of which considerably higher dispersion is required.

FIG. 4.—This exhibits the spectrum of the night-sky radiations in the red and infra-red. It will be noted that the strongest radiations are in the infra-red. This plate was not sensitive beyond λ 8700.

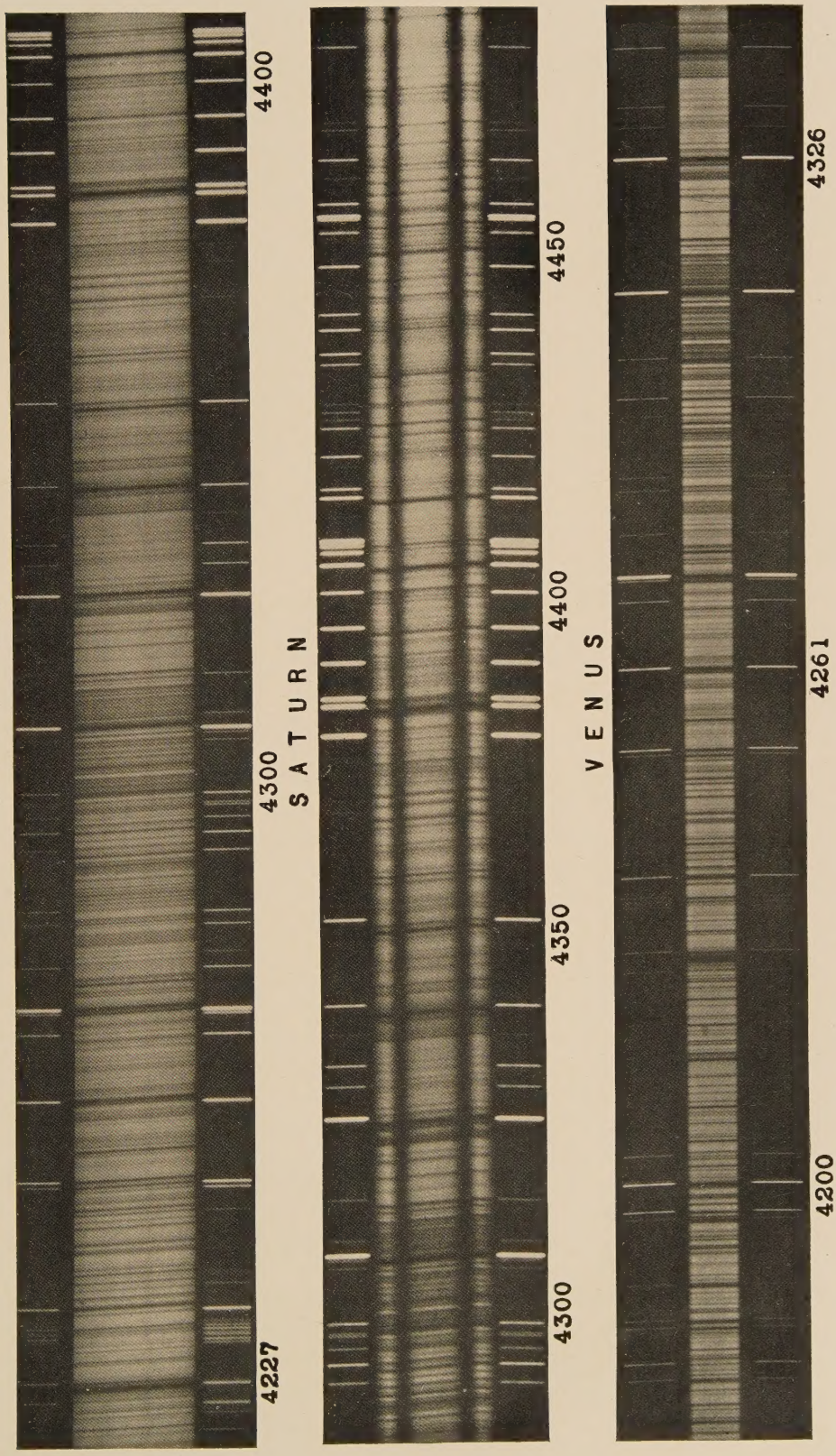
FIG. 5.—This shows a longer exposure on red radiations, and reveals the fact that there are a number of fainter ones throughout this range of spectrum.

PLATE 17

Here are shown three exposures for the spectrum of the night-sky, as they are usually made with the multiple image spectrograph. The chief interest here is to show that the negative nitrogen bands typical of the spectrum of an auroral display are usually present in the morning and evening skies for a short interval when sunlight is first or last touching the Earth's upper atmosphere. This is seen in the auroral bands near λ 3914 and λ 4278.

On the left of this plate will be seen a series of equally timed exposures made in recording the variation through the night of the chief auroral line at λ 5577. It will be noted that on this night this radiation was most intense during the evening hours and again in the morning, but this is not always the order of variation of the line.

To face Plate 13.]



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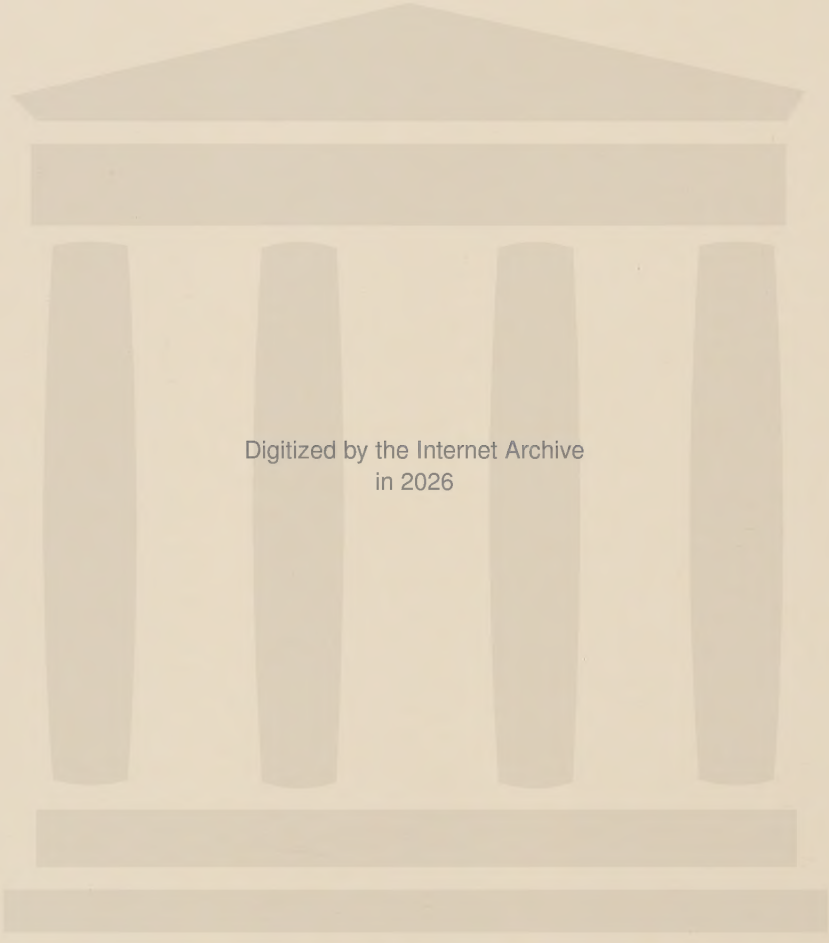

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Especially favourable and important have been these advantages in the study of such planets as we desire to test critically for water and oxygen, the bands of which are already in the spectrum due to the action of the air of the Earth.

The most precise procedure in the spectrographic study of the planet is to photograph alongside it that of the Moon at the same elevation in the sky. This furnishes a quite dependable comparison, for in this manner the paths in the Earth's air have been made exactly equal in the two spectra. The situation in such observations may be perhaps more definitely expressed as follows :—

Let C denote the observed spectrum of the comparison,
„ P denote the observed spectrum of the planet,
„ S denote the sunlight spectrum,
„ e denote the absorption of the Earth's atmosphere,
„ p denote the absorption of the planet's atmosphere.

Then we may write :

$$C = S + e \quad (1)$$

and

$$P = S + e + p \quad (2).$$

And hence $P - C = p$, the planet's own absorptions, where it is assumed the altitudes were the same for C and P, and thus making e exactly the same in (1) and (2). And when this is accomplished, the differences between the comparison and the planet can then safely be attributed to the peculiar action of the planet in question.

Fortunately at the outset there was available for this study as good a spectrograph, of the universal type, as the famous instrument makers, The John A. Brashear Co., could produce. This equipment in the years since has been added to as more efficient prisms or lenses became available for the requirements of the specific pieces of work. The adaptability of the original instrument made it exceptionally well suited to these planetary investigations, because various dispersions and various light-powers have been required in different phases of the work. For example, for the brilliant Venus one is able to utilize the highest available dispersion to advantage in testing for her rotation, whereas to study the infra-red spectra of Uranus and Neptune an instrument of low dispersion and a telescope of high light-power are required, owing to the extreme faintness of those spectra in the long wave-lengths, caused by their very intense atmospheric absorptions there.

From the first the best available commercial plates have been selected and they have been treated with sensitizing dyes whenever it has been possible to improve them for the task in hand. Generally the commercial colour-sensitive dry plates have a rather limited belt of colour sensitiveness, whereas the spectrum work needs a longer range of wave-lengths of more even sensitivity ; and nearly always considerable improvement could be made to the plates by treating them in the dark room with properly chosen sensitizers. In short, no pains have been spared to secure for this work the best possible photographic means.

It will be seen from the illustrations that were one testing for these

telluric bands, those at B and A of oxygen and the little *a* of water-vapour would doubtless be the best of all because of their great intensity. Consequently those have been made use of when testing for water or oxygen absorptions in Mars and Venus, because they would appear with only small amounts of oxygen and water-vapour present in the spectrum under examination. (See Plate 14.)

Mercury.—The spectrum of Mercury is not easy of observation in the blue and violet because of the planet's nearness to the Sun. But the less refrangible region is fairly accessible and has been observed at Flagstaff, although not as completely as can now be done with Dr. Mees' most recent infra-red plates. However, the spectrograms that have been secured of the planet reaching about to λ 9000 give no indications of dark bands peculiar to the planet even in the less refrangible region where such atmospheric bands are most apt to be found. This is not surprising in view of Mercury's nearness to the Sun, and its consequent intense insolation, which, combined with the planet's low surface gravity, doubtless long ago lost Mercury its atmosphere.

Since there has not been much question in regard to Mercury's diurnal rotation period, and to determine it spectrographically would require some special equipment for the task, I have made no attempt to do it.

Venus.—Because Venus has been so difficult telescopically it has received special attention spectrographically, at Lowell : (1) to observe for the planet's rotation, and (2) to search her spectrum for any evidence of atmosphere. I made the first of these studies in 1903, using the 3-prism spectrograph attached to the 24-inch Clark refractor and measuring, by the inclination method, the region from about λ 4200 to λ 4400. Fine grained plates were employed and the exposures made during the steady period of the early evening immediately following the setting of the Sun. The spectrograph was so designed as to allow of its easy rotation about the optical axis of the telescope in order that its slit might be given any desired position-angle. In particular, it was easily possible to rotate the spectrograph through 180° and thereby in the inclination method to reverse any rotation effect the planet might possess. This feature was regularly utilized in making the Venus spectrograms so that there were about an equal number in the two positions—direct and reversed—of the instrument, and also several with the slit inclined at 90° to these positions. The three kinds of plates were then shuffled, their original designations covered and were given, by an assistant, fictitious numbers in order that I could carry out their measurement without any knowledge of the plates or of the direction in which the lines might be inclined by diurnal rotation. The result of the plate-measures for the three positions of the instrument was practically a null value for rotation effects. The small mean differences between Venus and comparison lines were only of the nature of accidental errors of observation. The rotation effect was too small to be detected by the method under the conditions, although they were carefully selected to have them the most favourable possible.

Fig. 1

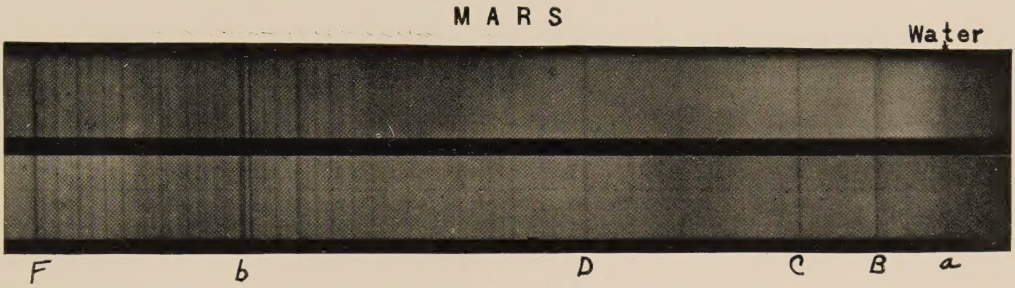

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Fig. 2

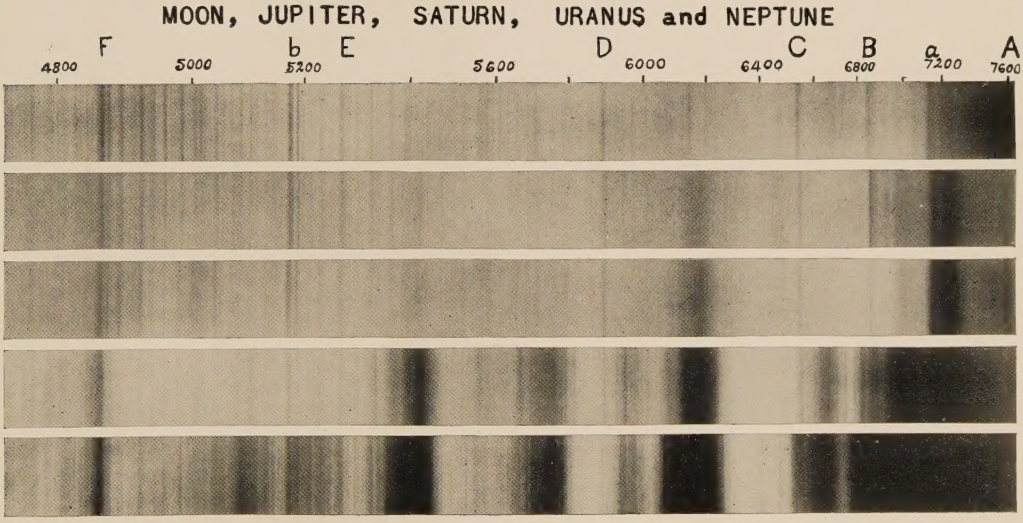
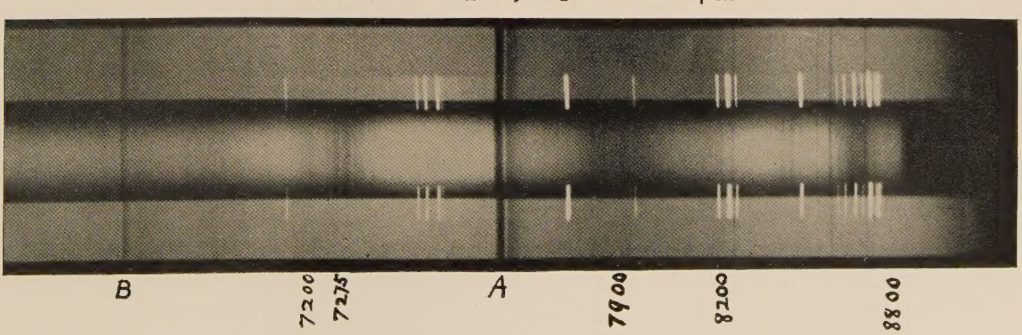


Fig. 3



Then, in order to determine with what accuracy the result might be possessed, I set about making a like series of spectrograms of Mars. These were treated in the same manner for measurement and resulted in finding for Mars a rotation period only about two hours longer than his true rotation time. This test of the accuracy of the Venus result implied that her period could not well be shorter than ten days or two weeks and fail to impress itself on the measures of the spectrograms, unless the planet's axis were, during the observing period, pointed nearly at the Earth. Other sets of plates have been made since to check for that possibility, but without detecting any certain rotational effects. Hence the rotation period of Venus would seem to be at least a few weeks long in order to satisfy the spectrographic tests, which quite definitely preclude the short period sometimes assigned to the planet in the past.

Spectrograms have been secured of Venus on several occasions in an attempt to detect any selective absorption due to her atmosphere. In 1905 plates were taken, following Lowell's suggestion that, if the observations be made at opportune times, the relative line-of-sight velocity could be made to shift from the Earth components her oxygen and water-vapour absorption lines, with her other lines, sufficiently to let them be disclosed. However, the negative result then got for both Venus and Mars by this method could not be definitive because the dispersion of the spectra was not great enough for the needs of the problem.

In 1908 plates of Venus were secured, with the Sun for comparison, under the exceptionally favourable conditions with Venus nearly in the zenith and the Earth's air very dry. However, the oxygen and vapour bands showed no more strength in Venus than in the equally high Sun's comparison. Later, notably in 1921, plates were again made during the drier air condition in the winter months when the vapour content of the Flagstaff air would be very low. While the tests were carefully made in all respects, again I found no vapour intensification. Thus, evidently the light from Venus had not encountered sufficient oxygen or water-vapour to give detectable absorption.

Since then spectrograms have been made of Venus on the new infra-red plates, recording the spectrum beyond λ 10,000, to see if any bands might be found in this new region. In particular, I thought it of interest to see if any traces might be found of those bands that are so strong in the giant planets, but I found no evidence of any of them. My infra-red spectrograms are prismatic and of rather low dispersion, and I missed the slender absorptions recently found by the Mt. Wilson observers and identified as being due to carbon dioxide.

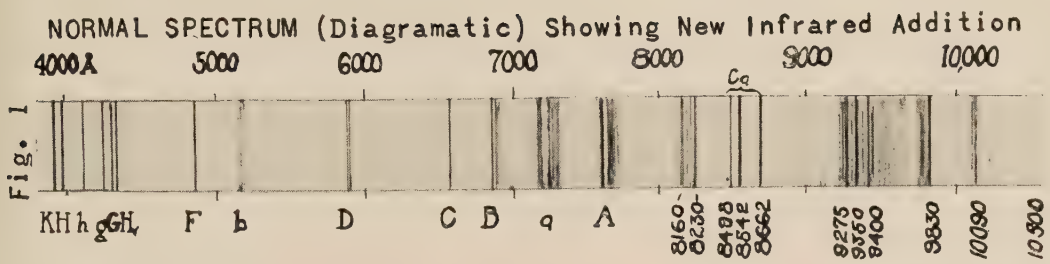
Thus the spectrum of Venus has not revealed to us much of the secrets of this very difficult planet. In view of her telescopic appearance it is rather surprising that she has no absorption bands in her spectrum. There is a hint, however, that careful tests might reveal something in the violet since it is possible to photograph shadings on her disc in light of that colour, but such bands might be expected to be wide and diffuse, and so difficult of detection unless one employs photometric measures.

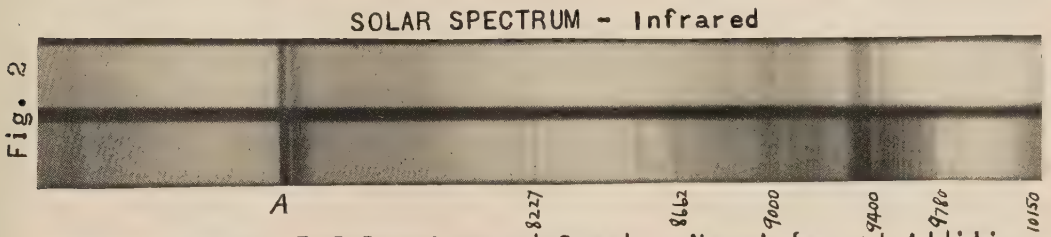
Mars.—Mars, long an object of peculiar interest at Lowell Observatory, has naturally been the subject of repeated observation with the spectrograph in seeking what evidence might be found on the much-debated question of the absorption bands of water-vapour and oxygen in his atmosphere. These were not secured under sufficiently exacting conditions to give definite evidence until the series obtained in the cold dry weather in the winter of 1908. The spectrograms of that series were made on plates sensitive to the red, including the very strong water-vapour band near λ 7250, which meant this favourable band could be employed as a test, and would be more capable of revealing small amounts than the weaker rain-band near D, which had been employed in previous tests of this kind. About a year earlier a number of sensitizing dyes had become available and by experimenting with these I found a combination of them, which, used on specially chosen plates, gave almost even sensitivity to λ 7200 in the red and with full exposure reached as far as to the oxygen band **A**. With these promising new plates it was realized that the best possible observing conditions in every way ought to be chosen. Simultaneous tests on the top of the San Francisco Mountains—almost a mile higher—compared with like tests at the observatory for altitude effects and others made to test the gain in dry winter air over dry summer air, showed clearly (*a*) that the mile higher Mountain Station gave less intense water-vapour and oxygen bands ; (*b*) that the driest *winter* air at the observatory gave very much fainter water absorption than the Mountain Station gave in favourable *autumn* weather ; and (*c*) that under the best *winter* conditions at the observatory, the chief vapour band is almost *invisible.* Thus the air condition is much more important for the moisture tests than the gain from the additional altitude of the mountain-top. Hence the plan adopted was to select very carefully, by tests for the purpose, the driest *winter* nights at the observatory for making the Martian observations.

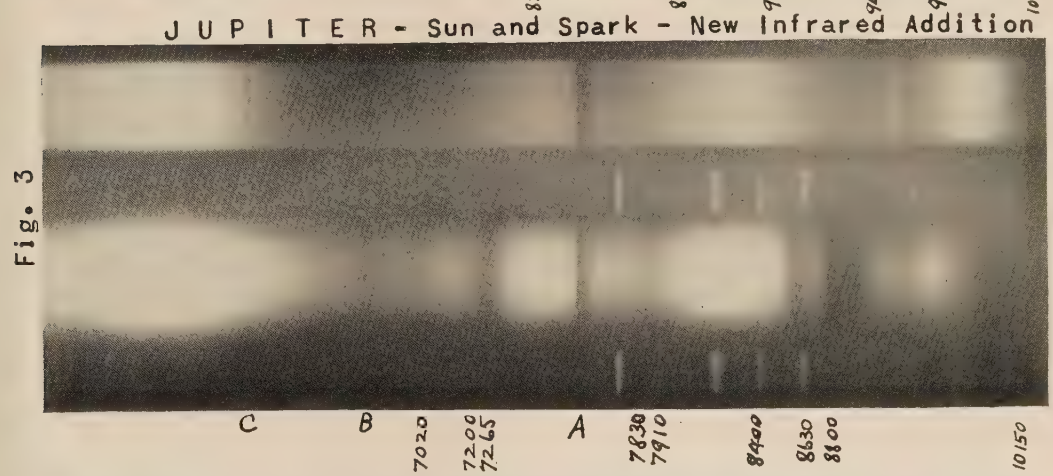
During the winter of 1908 the best nights were utilized in making spectrograms of Mars and the equally high Moon. On one of the most favourable nights for the plates of Mars, a long series of spectra of the Moon was also made from near the meridian to near the horizon to get the growth in the vapour band with increase of air path. From the illustrations of the Mars-Moon spectrogram of that night and of one of the three Moon spectrograms it should be possible to see the increase in the vapour band in the Martian spectrum over that in the Moon, and the growth in this band with increasing zenith distance. It was such observational evidence as this that was the basis of my conclusion that the spectrograph had given competent proof of water-vapour on Mars.

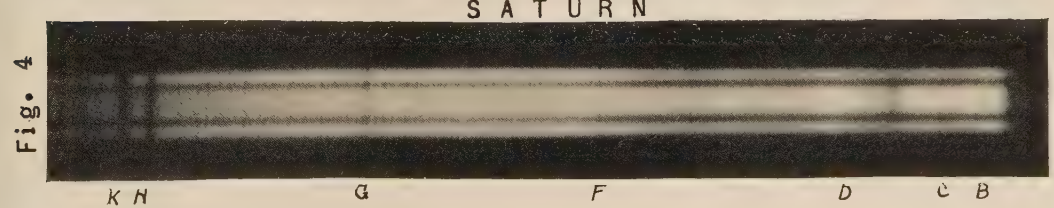
Afterwards the late Professor Very made a study of these plates and some similar ones I made in 1914 for quantitative values of the intensification of the Martian bands of water and oxygen, which also resulted in showing a quite definite, though small, accentuation due to the Martian atmosphere. In 1925, tests by Drs. Adams and St. John confirmed this result by securing small but comparable intensifications of the Martian bands.

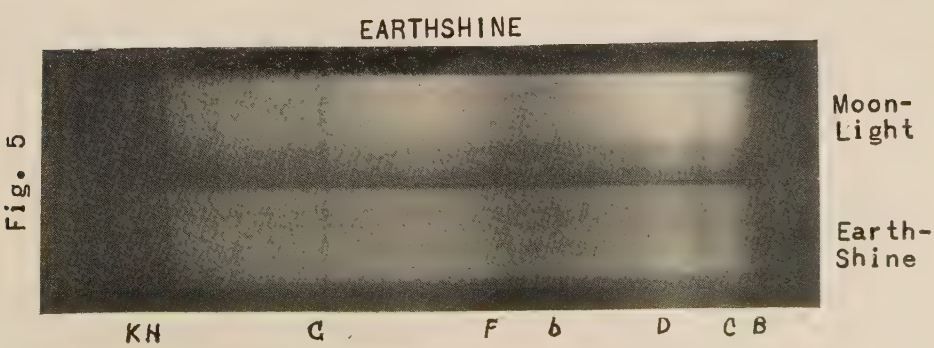
The spectrum of Mars has more recently been examined in the infra-red for still other absorptions, but none have as yet been detected. But with the











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recent strides made in developing efficient infra-red plates it is quite possible to improve the analysis of this important region of the spectrum, especially with higher dispersion.

Jupiter.—In passing to the next planet—Jupiter—it is well to note that we take two Bode planetary steps in one and over a gulf of difference between two very unlike groups of planets.

The brightness and large disc of Jupiter are great aids to his spectral study. His diurnal spin is strikingly displayed by his spectrum and its value may thereby be measured with high accuracy. (See Plate 13.)

The detection of weak diffuse planetary bands may be effectively done, usually with low dispersion, hence rather slow red-sensitive plates have in the past been useful in exploratory observations. An important advance was made into the red in 1907 when it was possible with new photographic sensitizers to record effectively the spectrum to the A band of oxygen. New Jovian bands were then revealed—the most prominent being the double near λ 7200–7265.

I then sought to determine what spectral differences might appear by comparing the spectrum of the strongest cloud belt with that of the bright equator and found only moderate or slight differences. The green region of the cloud-belt spectrum was much *weaker* than that of the *equator*. Some of the darker absorption bands like λ 6190 and λ 7200–7265 were a little stronger, but more notable was the case of greater equatorial strength of the composite band near λ 6460 as compared with its cloud-belt strength. This band seemed of interest out of proportion to its strength because it was very weak or wanting in Saturn's spectrum. It was observed with 3-prism dispersion and a number of its line-like components measured for wave-lengths. To these we shall have occasion to refer later when concerned with identifications.

The next forward step came with the new sensitizing dye Neocyanine, for it brought under study another 1000 angstroms of the infra-red region. As before, new absorptions were charted—the stronger of them were those near $\lambda\lambda$ 7920, 8400, 8615, and one (very black and wide, for it cut off sharply the spectrum) at λ 8810.

Finally, about a year ago with the advent of Dr. Mees' new deep infra-red plates it became possible to extend the record of Jupiter's spectrum to beyond λ 10,000, opening up an uncharted region far removed from the visible spectrum and of general interest apart from the problem immediately in hand. It has disclosed a number of new absorption bands and displays a disposition of the selective absorption to become almost general and complete.

Saturn.—Clerk Maxwell demonstrated that the rings of Saturn could not be a solid body and withstand the planet's pull; Keeler forty years later with the spectroscope showed them to be particles or small masses streaming around the planet in accord with theory. (See Plate 13.)

The spectrum of Saturn is so nearly like that of Jupiter that we need not

take time to do more than draw attention to the points in which they differ. It should be stated in passing that because the two are spectrally much alike not so much time has been given to observations of Saturn's spectrum in the infra-red. The band near λ 6460 in the spectrum of Jupiter is much weaker in Saturn, the wide composite band centering near λ 7920 has some more intense features in the latter planet, but the general resemblance of the more prominent bands is fairly close. The spectrum of the rings shows none of the absorption bands: this accords with the rings being without atmosphere.

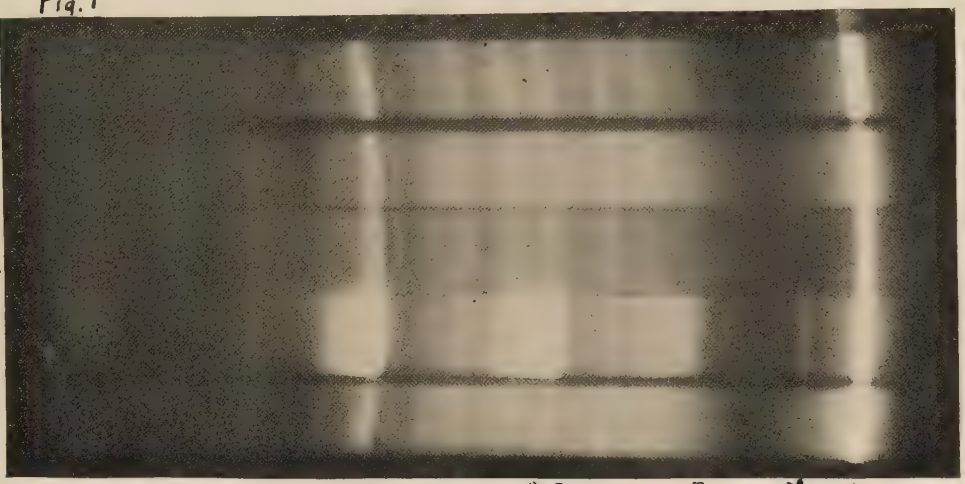
Uranus.—In 1911 the spectrographic investigation of the rotation of Uranus was completed after several years of work on the subject. The earlier plates obtained were hardly capable of giving a satisfactory result, because the image on the slit of the spectrograph, at the focus of the 24-inch telescope, was too small to give a wide enough spectrum to be conducive to accurate measurement of the inclination of the spectral lines. Then by interposing an amplifying lens giving an equivalent focal length of about 55 feet there resulted a well-measurable spectrum. The observations yielded concordant results, a quite reliable rotation period for the planet of 10·7 hours. The rotation was retrograde, as was rather to be expected from the orbits of his satellites. In this investigation half the plates were in the usual position of the spectrograph and the others with the spectrograph rotated 180°. Thus, of the plates half showed inclination in one direction, the other half in the opposite direction, which greatly facilitated the accuracy of their measurement.

In the study of its atmospheric absorption bands the Uranian spectrum has been reobserved as often as improvements in plates have made it possible to extend the work. These were the first photographs of the spectrum in the region from λ 5000 to λ 6000, although this portion had earlier been observed visually by Huggins, Keeler, Vogel and possibly by other observers. And for the first photographs of the blue and violet of the spectrum of Uranus we are indebted to Huggins, who showed that the planet had no dark bands in that portion. My first plates of the green and yellow exhibited no bands that had not been seen visually. But the first photographs of the red revealed some new bands, and others, more recent, reaching still farther into the less refrangible region, even as far as λ 8500, revealed still additional absorptions and made it clear that these continue to increase in number and intensity into the deep infra-red. There is evidence that almost all the light of longer wave-lengths is absorbed by the planet's atmosphere. How far this continues is not known, but the strength of the absorptions in this portion of the spectrum of Jupiter indicates for Uranus an even greater atmospheric absorbing power.

Neptune.—The spectrum of Neptune has been studied in the same way as that of Uranus ; indeed, when it was possible with new plates or new dyes to extend the spectrum of Uranus, it was usual to use them also for Neptune. Both planets have, therefore, been studied together, and it is interesting to note that their close resemblance has been striking. But of course Neptune

SPECTRAL RADIATIONS of the NIGHT SKY

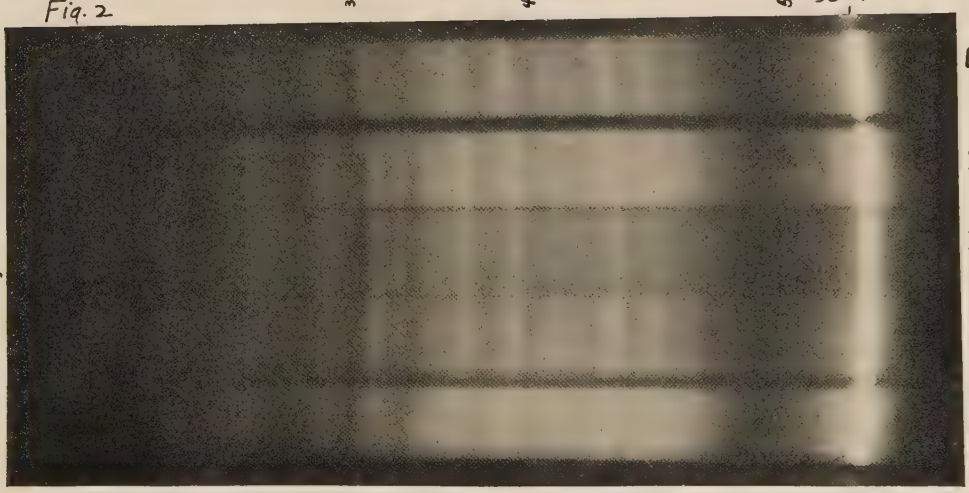
Fig. 1



Aurora May 29, 1932

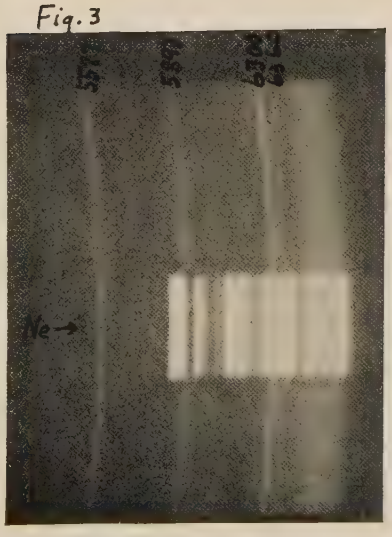
3914 K H 4278 G F 5206 5577

Fig. 2



Normal Sky Radiations

Fig. 3



RED and INFRARED RADIATIONS

Fig. 4

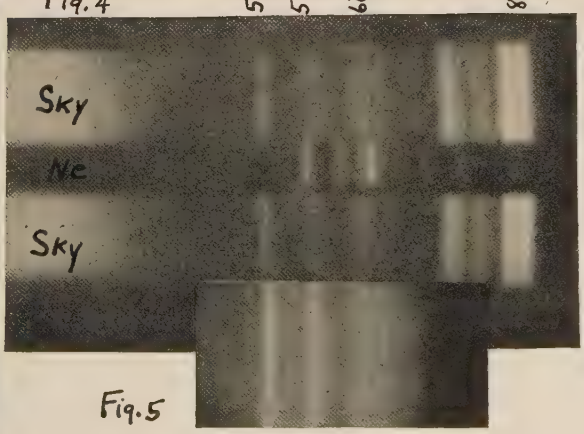


Fig. 5

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has shown always, in each new region, all the bands found in Uranus, in somewhat greater intensity, and also some new weak ones. Their increase in number and darkness into the longer wave-lengths is even more marked than in the spectra of the less distant planets. A plate exposed last year for the infra-red all but failed to show any trace of spectrum. As its exposure was long enough to have recorded some light, it appears likely that the weakness it showed in the infra-red is a result of the increased absorption of the planet's light in that region. Unfortunately we have not yet been able with the deep infra-red plates of Dr. Mees to get the spectra of either Uranus or Neptune beyond λ 8600, as can be done with the brighter planets. The recording of the infra-red spectra of Uranus and Neptune is far more difficult than would be inferred from the strength of the blue portion of their spectra.

Pluto.—Pluto is too faint for study of its spectrum with our equipment at Flagstaff. Instead, we early sought to learn something of the colour of its light by comparing it and near-by stars visually at the 24-inch refractor, with prints from plates of it and the same stars made with the 13-inch telescope. It soon appeared evident that Pluto was brighter visually than photographically. Then Dr. Lampland made exposures to the same end, with the 42-inch reflector, through an orange filter on panchromatic plates and on blue plates without filter, for a check. These showed Pluto to be brighter relative to most stars through the orange filter and *vice versa*, fainter on the blue (check) plates. He made at the same time similar orange filter and blue plates of Neptune and near-by stars, which showed Neptune as a blue star, as would be expected from the strong absorption of his yellow and red light by his atmosphere. Thus Pluto's light in the yellow and red is apparently somewhat stronger than normal sunlight, and so he is a yellowish planet, possibly standing between Mercury and Mars in colour. Higher precision is attainable in these colour tests, and the observations are being repeated with that objective in view.

It is quite beyond the spectrograph to deal with the diurnal rotation of Pluto. But this is not quite hopeless of detection by photometric study if the reflectivity from different faces of his disc differ enough to give an appreciable light variation, and the matter is having attention at the Lowell Observatory.

Earth.—Passing now from the most remote member of the solar family let us consider for a few minutes our own planet. If we could observe the Earth from a distance, as we do the other planets, we should doubtless learn some valuable facts. The spectrum of its daylit side would be valuable for comparisons with the spectra of the other planets. The best substitute for this idealistic observation is one of the earthshine on the Moon. Its spectrum makes it clear that the Earth is of bluish colour. (See Plate 15.)

The night side of the Earth is not quite dark but faintly luminous. This faint illumination has been under study spectrographically at Flagstaff since 1915, and has been found to analyse into some very interesting spectral radiations. By the autumn of 1916 the unfailing spectral records of the

so-called auroral line in the yellow made it quite a patent fact that this light source was, as I then announced, permanently present in the night sky. It showed stronger toward the horizon and in the part of the sky toward the Sun. I found the wave-length of the chief auroral line to be the same in this permanent diffuse auroral light as in the spectrum of light of auroral displays; but that the negative nitrogen bands typical of auroral display spectra do not accompany this chief yellow line in the sky, except when auroral displays are actually present. However, these nitrogen bands, I found, could be photographed in the morning and evening skies if brief exposures were made at the moments when the first and last traces of sunlight touch the high atmosphere. Thus the day, as it were, begins and ends with a sort of auroral flash. (See Plate 17.)

In addition to the chief auroral line there are present a great many other radiations in the spectrum of the dark part of the night. These are distributed throughout the spectrum from the ultra-violet to as far as λ 8600 * in the infra-red, where appear the strongest of these, which doubtless much outshine the famous chief auroral line in the yellow. With two-prism dispersion it has been possible to measure with some approach to accuracy two lines in the red at λ 6300 and λ 6360, which undoubtedly are the two forbidden oxygen lines at λ 6300 and λ 6363. The wave-lengths of the other radiations, especially those in the infra-red, have not yet been determined with sufficient accuracy for their identification. The illustrations of these observations reveal some new facts concerning the optical nature of the night sky and the Earth's atmosphere. (See Plates 16 and 17.)

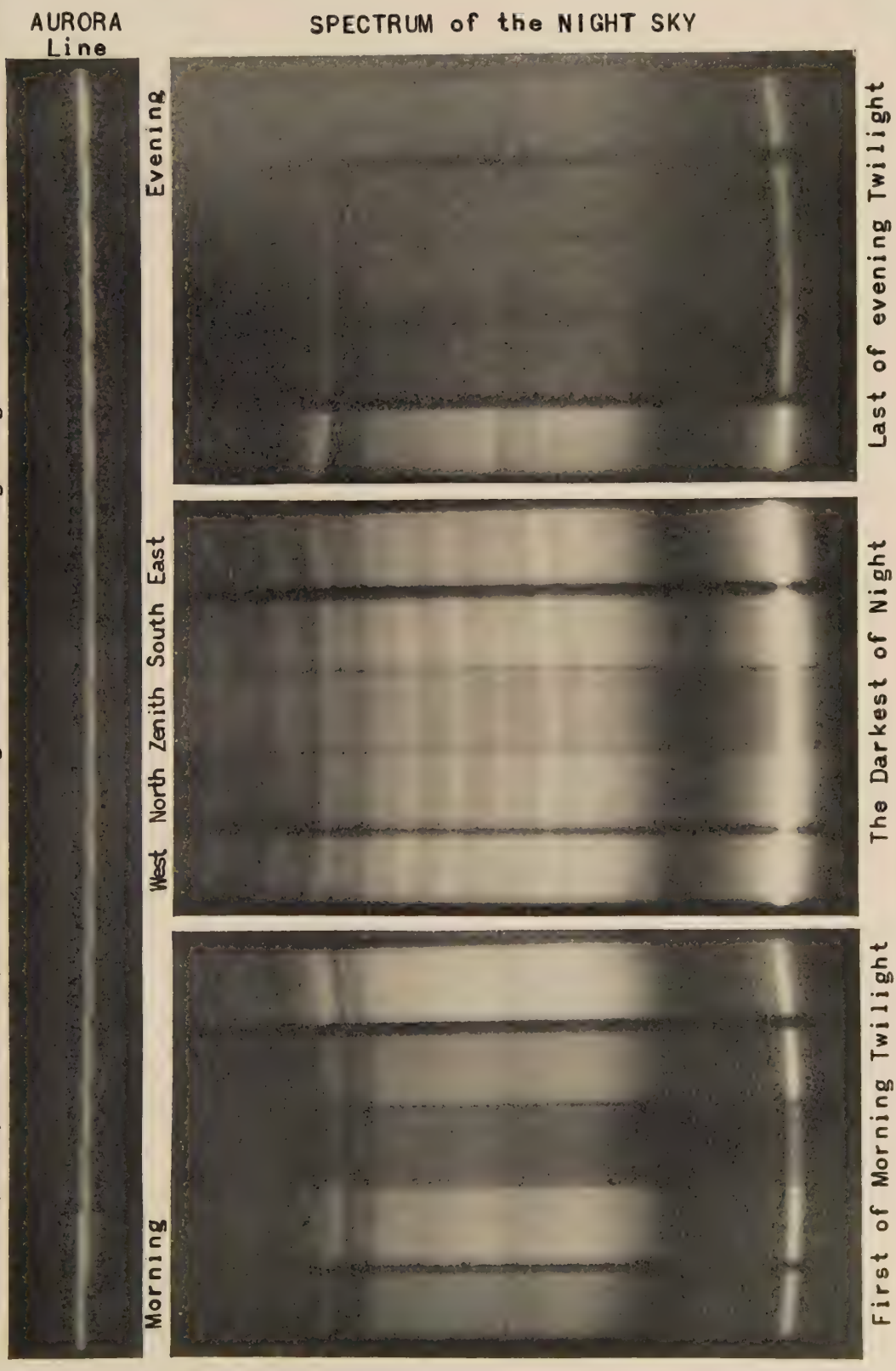
The known high albedos of the giant planets would seem to point to gases rather than to liquids or solids as the more probable agency to account for the dark bands.

The recent rapid development in the study of molecular spectra, especially in the less refrangible wave-lengths, offers promise that we may soon make great strides in the identification of the constituents concerned in the dark bands produced by the atmospheres of the outer planets.

About a year ago Dr. Wildt, of Göttingen, called attention to the fact that some strong lines in the NH_3 (Ammonia) spectrum coincided with certain strong lines observed in the spectrum of Jupiter, at Flagstaff, as follows: $\lambda\lambda$ 6450, 6456·6, 6473·8, 6500 and 6508. He also noticed that the bands of methane gas resembled some of those in Jupiter and Saturn. More recently Dr. Dunham, at Mount Wilson, obtained with the 100-inch reflector high dispersion spectra of Jupiter and Saturn, and compared them with the spectrum of NH_3 (Ammonia) in a 40-metre tube, showing a very clear identification. He has published some results of his work and kindly sent me some copies of his spectra with permission to use them on this occasion. He found NH_3 to have numerous lines also in the Jupiter and Saturn band near λ 7920, and he noted some resemblances in the methane and planetary bands.

The great range of new spectrum that has been added in the less refrangible region makes it urgent that a study be made of the deviations of

* The records have since been extended to 10,000, and additional radiations found.



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planetary spectra from the solar spectrum, at least at its ends. This will amount to finding albedos, as it were, for the violet and for the infra-red. The fact that one planet is blue and another is red makes it plain that their albedos for different widely separated colours must greatly differ. And since the infra-red region now observable comprises as great a span of wave-lengths as the whole of the visible spectrum, it goes without saying that our visual albedos are not now usable for the infra-red. Something has been done in this direction, and we hope to carry it to a usable completion as soon as possible.

Summary and General Results

These observations of planetary spectra suggest that the planets fall into two main classes: those which show little or only earth-like atmospheric bands, and those which show the strong selective absorptions which are peculiar to the giant planets. The first is the terrestrial group, Mercury, Venus, Mars, Pluto and the Earth. To the second group belong the four giant planets. Pluto, as well as present observations can decide, belongs oddly to the terrestrial group, although his distance relation with Neptune would have suggested otherwise. It has generally been concluded that distance from the Sun is the main factor involved in the outstanding general increase in the absorption bands from Jupiter to Neptune. But Saturn's absorption bands are hardly appreciably stronger than those of Jupiter, whereas, if the increase is dependent upon the distance from the Sun, the difference between these two planets should be as great as for any step of the series. And it is hard to believe that the atmospheric disturbance of Saturn occasioned by the very high inclination of his axis to his orbit could be a very powerful factor in his atmospheric absorptions, although, no doubt, this may be quite a dominant factor in smoothing out his *cloud belts* as compared with those of Jupiter. Might it be that Saturn's low density acts against his atmosphere's exercising as intense selective absorption as it otherwise might do? Venus visually has given by albedo, etc., indication of having a considerable atmosphere, and from her size, density and distance from the Sun caused us to expect to find evidence in her spectrum of oxygen and water-vapour, *i.e.* an earth-like atmosphere. Instead, even traces of these are absent, as shown by many tests at Flagstaff and also at Mount Wilson, where recently evidence has been found for carbon dioxide. Thus, Venus is still holding out to us an enticing problem, to test our scientific ingenuity and skill.

As was mentioned before, the great increase in selective absorption from the yellow into the red and infra-red from Jupiter to Neptune accounts for the bluish colour of Uranus and especially of Neptune. These two latter planets are faint for our available infra-red sensitive plates, but such spectrograms as I have been able to get of their infra-red regions leave little doubt that their longer wave-lengths are virtually wiped out by their strong atmospheric absorptions. Even the spectrum of Jupiter is very weak in the λ 10,000 region, and there, doubtless, Uranus and Neptune would have practically no light left. It is thus conceivable and apparently probable that this selective

and general absorption of the atmospheres of these giant planets continues deep into the heat spectrum, where it may be sufficiently strong greatly to reduce the heat radiations from these planets in the region to which our own air is transparent. Obviously solar energy taken in by these planets must be again given out, altered or unaltered, in order to maintain the steady state, but if the giant planet's atmosphere obstructs the spectrum range which our air transmits, and ours, *vice versa*, obstructs where the other transmits, we receive no radiations however much there may be. Here we probably have, I think, the reason the radiometric observations made at Lowell by Dr. Coblentz and Dr. Lampland (and later also at Mount Wilson), as interpreted by Menzel, indicate such low temperatures for Jupiter and Saturn. These spectral studies suggest then an explanation for the derived radiometric temperatures being so much lower for Jupiter than the vigorous activity of his surface markings would suggest. It seems that a planet's atmosphere may thus be a most potent factor in deciding what its temperature shall be.

In conclusion it is pertinent to make reference to the fact that Sir George Darwin, in whose memory these lectures are named, himself served for twenty years on the Council of the Meteorological Office. The present lecture therefore, concerned in the main with the atmospheres of the planets, has, I believe, touched upon matters in which he was directly interested. In bringing the lecture to a close I realize that many phases of the subject must appear in an unsettled state as a natural consequence of the development of science, when we are apt to raise more questions than we answer when we set out to solve a definite problem.

REMARKS ON WORLD-STRUCTURE.

E. A. Milne, F.R.S.

1. In recent papers H. P. Robertson * and W. O. Kermack and W. H. McCrea have obtained,† independently but by almost identical methods, the welcome result that a limiting case of the hydrodynamical model of the universe given by myself ‡ coincides with a limiting case of a series of universes given by the usual theory of relativistic cosmology, the limit in each case being that for which "the density tends to zero." § The same writers also make criticisms of my application of a certain postulate used by me, of my discussion of "cosmic time" and also (in the case of Kermack and McCrea) of my treatment of gravitation. I desire to acknowledge the sympathetic and helpful character of these criticisms, and if I reply to them

* Robertson, *Zeit. für Astrophys.*, **7**, Heft 3, 1933.

† Kermack and McCrea, *M.N.*, **93**, 519, 1933.

‡ *Zeit. für Astrophys.*, **6**, Heft 1–2, 1933.

§ A full treatment was given in a course of public lectures in the University College of Wales, Aberystwyth, 1933 March 6–10, and in various lectures at Oxford and elsewhere, since my *Zeit. für Astrophys.* paper was written.

I trust my reply will not be taken in any controversial spirit. In this note I also develop somewhat further the ideas of my recent paper along lines already given in recent lectures.*

2. McVittie, Robertson, and Kermack and McCrea have all pointed out that in constructing both my hydrodynamical and my statistical systems I made use of privileged observers, namely, a set of observers separating from one another (at $x=0$, $y=0$, $z=0$ in the reckoning of each observer) at a time $t=0$ (in the time-reckoning of each observer) with uniform relative velocities in straight lines.† I imposed the condition that every such observer must view the same world-history (*i.e.* see the same sequence of world-pictures). But none of the writers mentioned have noticed that the systems when once constructed possess a property which is independent of the mention of the privileged observers. This property may be stated as follows :—

The members of all pairs of particles in uniform relative motion belonging to either the hydrodynamical or the statistical systems possess indistinguishable world-views.

For in the hydrodynamical system all the particles, members of the system, are in uniform relative motion, and coincide with particles carrying the original privileged observers. They therefore possess indistinguishable world-views. And in the statistical system, by my acceleration formulæ, the only particles with zero acceleration satisfy the relations $x=ut$, $y=vt$, $z=wt$ and thus correspond to the originally introduced privileged observers. They also, therefore, possess indistinguishable world-views.

Although the postulate was loosely repeated in the body of my paper, it was stated in the small print summary at the beginning as applying not to all observers but only to observers in uniform relative motion. When we are studying systems capable of representing the whole universe, the systems constructed necessarily include the totality of particles in the model concerned, and thus it is illegitimate to employ any observers save those represented by particles which are actual members of the systems when constructed.‡ Conceptual observers not represented by occurring particles cannot legitimately be introduced. It must be remembered that the "privileged observers" do not automatically count themselves in as part of the system they are to construct. They are mere scaffolding which can be removed as soon as the systems have been constructed ; and it then becomes an *a posteriori* test whether, of the particles occurring in the system thus constructed, the members of every pair of particles in uniform relative motion possesses indistinguishable world-views. This test is satisfied rigorously by each of my two systems. The restriction of the equivalence property to particles in *uniform* relative motion is in accordance with the special theory

* The expression is that of Kermack and McCrea. But it should be noted that in the experience of any one observer there are always distant regions where the density takes arbitrarily large values, whatever the constant of proportionality in the density-formula. This point is obscured in Kermack and McCrea's formula (16)—one of the disadvantages of using cosmic time.

† Privileged observers are of course tacitly employed in the Einstein and Lemaître universes, as, for example, in assuming homogeneity.

‡ The same principle is implicit in the Einstein and Lemaître universes.

of relativity which was employed throughout. In the special theory of relativity absolute acceleration has a meaning, and relatively accelerated observers are not equivalent.

Thus privileged observers were, in fact, employed only as means of constructing the systems, and the systems themselves thus constructed are seen to possess the property stated above in italics. This I posit as the condition which a system must satisfy if it is to represent the universe. It is presupposed that the universe has been smoothed out either to a hydrodynamical or statistical system and that flat space and Newtonian time have been selected as the means of description. It does not follow from this, of course, that the systems so constructed are the most general systems satisfying the postulate as stated. Kermack and McCrea suggest, in fact, that more general systems could be constructed by rotating an arbitrary curve (as typical world-line) about a centre in Minkowski space-time. But unless the curves in question reduce to straight lines through the origin, there would be a discrete number (greater than unity) of world-directions through any point, and the system would be neither a hydrodynamical one nor a statistical one. Further, for the system to contain particles in uniform relative motion other than concurrent particles, the particles would have to be moving along skew straight lines, and then difficulties of time-origin would arise. In order to superpose the observed world-histories, for the purpose of testing their indistinguishability, a time-origin must be chosen in the experience of each observer, and this must coincide with the epoch of minimum volume of the system in the experience of any observer. If, then, $t = 0$ were chosen as the moment of closest separation of a given particle P_0 and each of a set of uniformly moving particles P_1, P_2, . . . this would not be the moment of closest separation of P_1 from P_2, etc., unless this closest distance were zero in every case. Until systems possessing the property given in the re-stated postulate are actually constructed—and it must be remembered that the equivalence-of-view property must extend to all epochs —it seems fair to suppose that the systems I have constructed are the most general ones.

(It should be borne in mind that the hydrodynamical model is the limit to which *any* kinematical system of particles in uniform relative velocity must tend when it possesses no preferential velocity-frame. For it is obtained asymptotically by making the asymptotic substitution,

$$u \sim x/t, \qquad v \sim y/t, \qquad w \sim z/t$$

in the invariant velocity-distribution

$$\frac{A du dv dw}{c^3(1 - \Sigma u^2/c^2)^2}.$$

This way of regarding the hydrodynamical system involves no privileged observers even in its construction ; for *all* frames in uniform velocity are made equivalent as regards the velocity-distribution.) *

3. Kermack and McCrea deny that the phenomena of gravitation can be

* Robertson, Kermack and McCrea all ignore this property.

obtained by my methods, for they claim that in my statistical system solutions corresponding to different functions ψ can be superposed without affecting the motions in the corresponding systems, thus implying zero gravitational interaction. But this wholly ignores my derivation of the acceleration formulæ which were the basis of the gravitational discussion; * these formulæ show that the accelerations corresponding to the superposition of two ψ's are quite different from those for either ψ separately. Kermack and McCrea indeed admit this in a footnote, but claim that I have "sacrificed the purely *statistical* character of the solution in assuming that a particle with given velocity and position, at a given instant, possesses a *uniquely* determined and not merely statistically determined acceleration." † But such a situation is part of the very definition of a spatio-velocity distribution. In a spatio-velocity distribution the accelerations can be descriptively enumerated as functions of t, x, y, z, u, v, w. It would be quite possible to follow up the implied suggestion of Kermack and McCrea and construct a *space-velocity-acceleration* distribution function. But the principle of conservation of particle-number would still lead to a generalised Boltzmann equation, which would in turn determine the rates of change of the accelerations p, q, r as functions of $t, x, y, z, u, v, w, p, q, r$. Each orbit would then depend on nine constants of integration, the "initial" values of $x, \ldots r$. I proved that for a spatio-velocity distribution the occupancy of space-time by the world-lines determines their curvature—an interesting and surprising result which seems to be equivalent to gravitation. It must be borne in mind that the law of gravitation to which such considerations lead is the large-scale law, describing only the accelerations remaining when a prescribed amount of smoothing has been carried out. Since in the smoothing-out process planetary detail has been necessarily obliterated, the method will not necessarily yield the Keplerian motions of planetary systems. It should, however, properly developed, yield insight into such non-Keplerian motions as are indicated by the forms of the arms of spiral nebulæ,‡ for these appear to be large-scale phenomena. I have in fact found the condition that my large-scale law of gravitation should agree with Newtonian gravitation at very small distances from the observer, and thus evaluated the constant C occurring in the acceleration-formulæ in terms of the Newtonian "constant" G. Robertson has claimed to have established the full equivalence of the statistical and hydrodynamical systems, but the integral he gives § is purely formal, as it can be shown to diverge. The full treatment of the relation between the hydrodynamical and statistical systems is interesting though complicated. I hope to publish the details on another occasion. (They were given in lectures at Oxford before Robertson's paper.)

* Kermack and McCrea substitute some other theory for mine, and then complain that their substituted theory does not yield gravitation. I am not responsible for the impotence of their substituted theory.

† It is not easy to see what Kermack and McCrea mean by a "statistically determined acceleration." I determined the actual acceleration of each particle if the number of particles was to be conserved.

‡ *Cf.* recent papers by H. Vogt.

§ Equation (2.11) of his paper.

4. When we smooth out the universe to a hydrodynamical system, and construct a model of it in accordance with the above postulate, the gravitational field reduces to zero. This appears to be true whatever the density,* though McCrea and Kermack doubt the validity of this result. I therefore give a formal proof for the hydrodynamical model. Consider a number of frames of reference, moving with the velocities

$$u = x/t, \qquad v = y/t, \qquad w = z/t$$

in the experience of any one of them, and distributed according to the density-law

$$\frac{A t dx dy dz}{c^3(t^2 - \Sigma x^2/c^2)^2}.$$

Place a particle at the origin of each frame and let each particle be instantaneously at rest in its own frame. It is required to determine the instantaneous accelerations of the particles relative to one another when each particle is acted on by all the others in accordance with any given law of gravitation. Consider a pair of particles P and Q in their respective frames F_P and F_Q. Then each in its own frame is central with respect to the whole system and so is unaccelerated in its own frame. But the frames are unaccelerated with respect to one another. Hence the particles are unaccelerated with respect to one another. Hence the resultant field is zero. Kermack and McCrea object that in the universes of general relativity each particle is still central yet accelerated with respect to other particles. Precisely, for there each particle is only central when referred to a frame *accelerated* with respect to other frames. In my system the frames in which the particles are central are in *uniform relative* motion, and the relative accelerations of the particles are accordingly zero.

5. In this hydrodynamical system, accordingly, the relative accelerations are zero whatever the constant A occurring in the density-distribution. Two or more such systems may be superposed, and since each had a gravitational field rigorously zero, so has the sum.

Robertson, and Kermack and McCrea, using the enormous power behind Einstein's field equations and making the additional homogeneity assumption usual in relativistic cosmology, have still only obtained a limiting case of my hydrodynamical distribution. That this limiting case corresponds to zero density in both my system and in the general relativity solution is inevitable, for the writers mentioned use a metric in the sense of general relativity, with its implied statement of certain laws of nature,† whereas I used Minkowski space-time merely as a map, with the laws of nature not prescribed but

* Actually, comparison with the average properties of the statistical system determines a definite non-zero value for the density. This value is connected with the rate of expansion by substantially the same formula as in Einstein and de Sitter's solution, as I showed in my paper.

† *Cf.* S. R. Milner, *Proc. Roy. Soc.*, A, **139**, 349, § 2, 1933. The laws in question are the field equations, which determine the distribution of matter in motion from the $g_{\mu\nu}$'s, and the geodesic equations of motion, which prescribe the paths of freely moving test-particles.

necessarily different from their formulation in general relativity. The two reduce to coincidence when the density of the matter tends to zero. But it is at no point necessary in my procedure to assume zero density ; whatever the density the Lorentz formulæ maintain their validity for observers in uniform relative motion when such observers are equivalent in relation to the matter.

In order to obtain their result the writers mentioned have to begin with expanding space of constant negative curvature, to assume *a priori* a homogeneous density-distribution and to assume that this density is a function of a certain introduced variable called cosmic time ; and they then find that particles of fixed co-ordinates in this space behave in the same way as certain moving particles in ordinary Euclidean space ! The advantage both of directness of statement and of economy of physical postulate would appear to lie with my method of deduction. But the work of Robertson and of Kermack and McCrea * is highly important as showing the conceptual character of the space-time of general relativity ; space-time is clearly no constituent of the world itself, but only a construct employed for the description of familiar experiences in unfamiliar terms. Those who prefer expanding space of constant negative curvature and who desire to discuss particles stationary in this space may legitimately do so. But they are at bottom only discussing moving particles in flat space. The disadvantage of general relativity is that it cannot employ flat space in describing gravitation.

It should be noticed that in order to compare the general relativity results for $\rho \neq 0$ with my results, either my hydrodynamic distribution would have to be mapped (for $\rho \neq 0$) in a space-time of general relativity, or else the general relativity universe would have to be mapped in flat space. Otherwise no comparison is possible. As far as I can see, Kermack and McCrea's discussion of the accelerations they calculate for $\rho \neq 0$ does not throw any light on the experiences the observer may expect in flat space until this mapping process has been carried out. Again, the writers in question use my co-ordinates x, y, z, t without any discussion of the zeros from which they are measured ; from their standpoint x, y, z, t are mere transformations of their co-ordinates τ, θ, ϕ, χ, and it is not shown how these co-ordinates with their implied zeros are to be translated into experience.† In my treatment, which began with postulates stated in terms of experience, these zeros were fully discussed, and in effect my critics take over my discussion. Lastly, my postulates involve the experience of an observer not only locally but in relation to the whole universe. Thus they are not logically equivalent to the homogeneity-postulate of relativistic cosmology,‡ which involves only the *local* experience of an observer. Indeed, until the time of a *distant* event in general relativity can be uniquely defined, in terms

* Similar results were suggested to me last November by O. Heckmann of Göttingen.

† It is not shown, for example, by them, how t is recognised as the time of experience for distant events.

‡ See Robertson, "Relativistic Cosmology," *Reviews of Modern Physics*, 5, 65, 1933 January.

of clock-readings taken by the observer concerned—and this has not yet been accomplished *—the world-history postulate cannot even be stated for a general space-time. This brings me to the question of "cosmic time," to discuss which I shall first develop a little further the properties of my hydrodynamic model.

6. *The World-map and the World-picture.*—In special relativity, given an observer carrying a clock and situated on one of the particles of zero acceleration, a unique time (by his clock) can be assigned to any distant event, and he can thus construct a world-map for any instant t by his clock. It is instructive, and it has practical astronomical consequences, to compare his "world-map" with his "world-picture," which is the snapshot of the universe he would make with a camera. The two differ owing to the finite time of travel of light. As far as I know, no one has yet calculated precisely the "world-picture" viewed by an observer in one of the universes of general relativity. This is what the observer actually observes, and constitutes his sole reality, as the world-map (in general relativity) involves a merely conventional assignment of a time co-ordinate. In special relativity, on the other hand, for a given model the world-picture is readily calculated, since the assignment of a time co-ordinate by an observer to a distant event is unambiguous.

Let a system of nebulæ, moving in accordance with the law

$$u = x/t, \qquad v = y/t, \qquad w = z/t, \tag{1}$$

and distributed according to the law

$$\frac{A t dx dy dz}{c^3(t^2 - \Sigma x^2/c^2)^2}, \tag{2}$$

be viewed by an observer at rest at the origin; t is the time by his clock. What does he record? Let a light-pulse leave a nebula at time t_1 and arrive at the observer at time t, all by his clock. Let the distance of the nebula at the moment of emission be r_1; this is the observed distance of the nebula. Then

$$t = t_1 + r_1/c. \tag{3}$$

Accordingly, if V is the radial velocity of the nebula,

$$V = r_1/t_1$$

$$= \frac{r_1}{t - r_1/c}, \tag{4}$$

and this should be the *observed* velocity-distance relationship. The graph of V against r_1 for a fixed epoch t should therefore be tangent to the linear law $V = r_1/t$ at the origin $r_1 = 0$, and then curve upwards. Since the maximum value of V is c the maximum observable value† of r_1 is $\frac{1}{2}ct$, which is accordingly the radius of the universe, *as observed* at time t. This is otherwise obvious; the radius of the world *in the world-map* at time t is ct, but the

* *Cf.* Etherington, *Phil. Mag.*, 15, 762, 1933.

† This was independently pointed out to me by Mr. B. M. Peek.

particles at just less than this distance, travelling as they are with a velocity just less than c, were at distance $\frac{1}{2}ct$ when they emitted the light which reaches the observer at epoch t.

Whether the observed extra-galactic nebular velocities will be found to obey (4) is a matter for future observation. The ratio of the velocities of two nebulæ at observed distances r_1 and $2r_1$ at epoch of observation t should be approximately

$$2\left(1+\frac{r_1}{ct}\right)$$

for r_1/ct small, where t is given approximately by r_1/V_1; the ratio should thus be approximately $2(1+V/c)$. The effect, if it exists, is possibly just observable within the present power of spectroscopes and telescopes.

Naturally any observed law of dependence of V on r_1 could be interpreted as a functional dependence of velocity on distance only, as in the general relativity universes. If, however, observation confirmed (4) it would be evidence in favour of *constant* velocities for the nebulæ, although the t in the denominator of (4) could not be distinguished observationally from a constant.

Similar modifications are required in any predicted density-law when expressed in terms of observed distances at any given epoch of observation t. We have, with the same notation, if r is the distance of the nebula at time t,

$$r/t = r_1/t_1,$$

whence

$$r = r_1\Big/\left(1-\frac{r_1}{ct}\right).$$

Hence

$$r^2dr = r_1^2dr_1\Big/\left(1-\frac{r_1}{ct}\right)^4.$$

Now (2) may be written as a distribution

$$\frac{4\pi Atr^2dr}{c^3(t^2-r^2/c^2)^2}. \tag{5}$$

This, the number of nebulæ between r and $r+dr$, may be expressed in terms of r_1 as

$$\frac{4\pi At}{c^3\left[t^2-\dfrac{r_1^2/c^2}{(1-r_1/ct)^2}\right]^2}\cdot\frac{r_1^2dr_1}{\left(1-\dfrac{r_1}{ct}\right)^4}=\frac{4\pi Ar_1^2dr_1}{c^3t(t-2r_1/c)^2}, \tag{5'}$$

which is accordingly the predicted distribution of nebulæ observable between r_1 and r_1+dr_1 at epoch of observation t. Expression (5′) gives the density-law of the *world-picture*, as against (5) which is that of the *world-map*. Expression (5′) shows again that the greatest observable value of r_1 is $\frac{1}{2}ct$. The interesting point about (5′) is that the departure from homogeneity in the world-picture is of the first order in r_1/ct, although the departure from homogeneity in the world-map is of the second order. In passing from

the world-map to the world-picture, we compress the outer portions of the system more than the inner, owing to the recession and the finite time of travel of the light. By (5′) the observed density near the observer should increase radially * according to the law

$$\rho = \rho_0\left(1 + \frac{4r_1}{ct}\right). \tag{6}$$

Owing to the large factor 4 this should give an observable effect inside the present range of observation. It can be shown that a formula of type (6) is not peculiar to my density-law (5), but holds approximately for any locally homogeneous world-map of particles undergoing recession according to an approximate velocity-distance proportionality.†

It may be emphasized that formulæ (4) and (5′) describe the *experiences* of an observer in my hydrodynamical model. It is much to be wished that exponents of other world-models would similarly specify the resulting *world-pictures* as presented to an observer capable of measuring distances and velocities and of making counts of nebulæ.

7. *Time-relations and Evolution.*—We may suppose each particle or nebula in the hydrodynamical model to be provided with a clock, the clocks to be synchronised so as to read $t = 0$ at the moment of closest approach.

Then it follows from the Lorentz formulæ that in the world-wide instant t of one given observer, the clock on a nebula receding with relative velocity V reads t', where

$$t' = t(1 - V^2/c^2)^{\frac{1}{2}}. \tag{7}$$

Thus in the world-wide instant t of any given clock, all the clocks in the universe read other times. None of these times can be preferred to any other. The "age" of the system reckoned from the epoch of closest separation, in the world-wide instant of any given observer, is different for any other observer. The system accordingly has no absolute age at any given event, only an age when an observer is specified who observes that event, and the age assigned depends on the observer chosen. Thus just as the system has no radius, so it has no age, only a radius and an age when a particular observer is specified at a particular stage of his experiences.

We may, however, go further. Since all the particles are equivalent, the experiences of each particle near itself will be similar. Hence the evolutionary history of each nebula, reckoned in its own time-scale, will be the same. Let now t_0 be the epoch of a recognisable evolutionary event in the history of a nebula, reckoned by the clock at its centre. Such an event might, for example, be the development of a particular stage in the spiral arms, or a particular degree of resolution in the nucleus. t_0 is a constant for any given evolutionary stage, on whatever nebula it occurs; it is characteristic of the evolutionary stage in question. Then by (7) we have, since here t' is t_0,

$$t = t_0(1 - V^2/c^2)^{-\frac{1}{2}}. \tag{8}$$

* *Temporally*, the density near the observer *decreases* as t^{-3}.

† The observational satisfaction of this test would be a proof that the re-shifts indicated genuine velocities.

It follows that however small is t_0, given t (our present epoch) we can always find a value of V so near c that the nebula with this velocity is, *in our present* (in our instantaneous world-map), now experiencing the event specified by t_0. The epoch T, in our experience, at which the light from the event t_0 actually reaches us, is given by

$$T = t + r/c,$$

where

$$r = Vt.$$

Accordingly,

$$T = t_0\left(\frac{1 + V/c}{1 - V/c}\right)^{\frac{1}{2}}. \tag{9}$$

This gives the relation between local time t_0 of the evolutionary event we are now witnessing and our own epoch of observation T. t_0 is the time of the clock on the distant nebula as read by a telescope at ourselves at time T. It again follows that given an evolutionary stage t_0 and a present epoch of observation T we can always determine a velocity V so that we are now witnessing on the V-nebula the event t_0. t_0 can be assigned any value between 0 and T. It follows that the aggregate of nebulæ observable at any one instant should display all possible stages of evolution, from the latest which is our own, for nebulæ near ourselves, to arbitrarily early stages, for the most distant, most swiftly moving nebulæ. For V all but equal to c, the nebulæ are all but invisible, t_0 is almost zero, and these nebulæ should exhibit a stage of evolution little removed from that corresponding to "creation," $t = 0$. Thus the whole past history of the universe back to $t = 0$ should be written for us in the heavens in the various evolutionary stages of the various nebulæ.

Speculative though these conclusions may appear, it seems difficult to avoid them if the special theory of relativity is to be trusted. Thus observation of the most distant nebulæ, in the future, should produce definite evidence for or against the clock-behaviour consequences of the special theory of relativity. This adds enormous force to the case for pushing telescopic power to the utmost limits. Whether similar results hold good on other cosmological theories depends on the meaning of "cosmic time," which I now proceed to discuss in the light of the interesting treatment due to Robertson and to Kermack and McCrea.

8. *Cosmic Time.*—Kermack and McCrea write: "Milne uses the space-time of special relativity described by the metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2, \tag{10}$$

whilst the space-time for a homogeneous isotropic universe in general relativity is

$$ds^2 = dt^2 - [R(t)]^2 d\sigma^2. \tag{11}$$

The spatial section of (10) has a Euclidean metric

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{12}$$

whilst that of (11) has a metric proportional to $d\sigma^2$. Now by a known

theorem the manifold of constant curvature $d\sigma^2$ can always be represented conformally on the Euclidean manifold (12)." Both Robertson and Kermack and McCrea then proceed to introduce a variable τ given by

$$\tau^2 = t^2 - x^2 - y^2 - z^2,$$

and identify τ with cosmic time, *i.e.* with the "t" of (11). It at once follows that the t in (11) is not the t used in (10). Hence when in the limit $\rho \to 0$ and (11) is transformable into (10), the section of the world "t = constant" is a totally different section from the section "τ = constant," or "$t^2 - \Sigma x^2$ = constant." "t = constant" is a spatial section of the world as experienced by the observer using Euclidean space. But what Kermack and McCrea call a "spatial section" of (11) is not an experienced spatial section of the world at all. In mapping the one on the other, Kermack and McCrea are mapping different collections of events on one another. The section "τ = constant" is a sort of temporo-spatial section of the world, obtained by considering each particle at a different epoch in its career, as reckoned in the experience of a given observer. This is not at all what we mean in experience by a spatial section of the world, yet in current interpretations of relativistic cosmology sections "τ = constant" (surfaces of constant cosmic time) are always treated as space-sections of the world, for example, when it is said that the world is homogeneous.

As an example of the difference of interpretation, we can easily show that while the experienced space-section of my hydrodynamical system, "t = constant," is of finite radius (namely, ct), the section "τ = constant" of the same system is of infinite radius. For τ = constant gives

$$t^2 - \Sigma x^2/c^2 = \text{constant},$$
$$\Sigma x^2/c^2 = V^2 t^2/c^2,$$

and so the distance of the nebula from the observer, namely, Vt, is equal to

$$V\tau\left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}.$$

For τ = constant this is indefinitely large for V indefinitely near c. Each nebula or particle is in effect transferred to such a distance that the local density at itself is equal to a certain given value. The more distant, more swiftly moving particles, in regions of high density at a given instant t, must be displaced by a very large amount before the density is sufficiently small. Thus it is totally misleading to treat events "τ = constant" as constituting a spatial section of the universe in any observer's experience. The variable τ is the quantity t_0 of § 7. Events "τ = constant" are also events "t_0 = constant." Thus events "τ = constant" correspond to the same stage of evolution; in particular to the same local density, measured locally.

Kermack and McCrea say: "Milne asks which observers actually keep cosmic time. The answer to his question is that the cosmic time of an event is the proper time of that one of his fundamental observers who experiences the event." It will be seen that these writers have not answered my question. I did not ask: "What is the cosmic time of an event?"

but "Which observer keeps cosmic time?" The answer is that no observer keeps cosmic time. An observer's clock keeps the time t, not τ, and t is the time assigned by him to any event; t and τ only coincide for events at himself. Time is something kept by a clock, and the co-ordinate τ is not the time of experience. Kermack and McCrea in effect admit this. They say "no co-ordinate has the physical properties of time applied universally." This is true of the co-ordinate τ, but not of t, which is precisely the ordinary time of experience applied universally. All observers in my model keep different times t; τ is not a time at all. The "surfaces" $\tau =$ constant are tangent to the surfaces $t =$ constant, but whilst the surfaces "$t =$ constant" include all events simultaneous in the experience of the observer concerned, the surfaces "$\tau =$ constant" are not simultaneous events for any observer whatever. To speak of an absolute simultaneity, as Eddington does in *The Expanding Universe*, is misleading.

It must be mentioned that my objections to cosmic time are not objections to the use of a preferential co-ordinate, of which I fully admit the legitimacy. My objection is to the calling of this co-ordinate a time, and to the assigning to it the properties of time. Kermack and McCrea definitely adopt this usage when they describe "$\tau =$ constant" as a "spatial" section of the universe. It is a 3-way section, but not a space-section of experience.

I think the discussion of Robertson and of Kermack and McCrea most valuable, because whereas in a Lemaître universe, for example, it is difficult even to define simultaneity as used in ordinary astronomical practice (as when we speak of a nova outburst as occurring so many years ago), in the limiting universe discussed by the above writers, which coincides with a limiting case of my hydrodynamic model, we are in a position actually to compare cosmic time τ with experienced time t, and thus to see that they are totally different. I feel that this clarification justifies my criticism of cosmic time.

In general relativity the surfaces of constant cosmic time τ have the following physical significance: Assuming a hydrodynamical system of flow, as usual, we may consider the corresponding family of geodesics in space-time. Fix a value ρ' of the density ρ arbitrarily, and determine the point P on each geodesic at which the local density, measured locally, is equal to ρ'. The locus of points P for constant ρ' is a 3-way section of the universe, and the family of these "surfaces" is a family of surfaces of constant density. This homogeneity is simply a matter of definition. It is now assumed that the proper-time interval along any two geodesics between the neighbouring surfaces $\rho = \rho'$ and $\rho = \rho' + d\rho'$ is the same. The surfaces "$\rho =$ constant" then coincide with the surfaces "$\tau =$ constant," where τ is a parameter measuring proper time along any geodesic. These surfaces are the surfaces of constant cosmic time.

Thus the real content of the so-called homogeneity assumption is that the *local* experiences of any two particles are assumed the same. This shows up at once the difference of content between this local-experience postulate and my postulate, which equates the *world-wide* experiences (either their world-maps or their world-pictures) of any two particles in uniform relative

motion. My postulate has thus a much bigger content. It is due to this that my postulate by itself has consequences which, as shown by Robertson and by McCrea and Kermack, coincide in certain cases with the consequences of adding to the "local homogeneity" postulate the dynamical content of the equations of general relativity.

9. *Summary.*—Recent criticisms of my world-structure work by Robertson and by Kermack and McCrea are in part accepted,* in part answered. The equivalence-of-world-view postulate is restated; the systems already constructed are shown to obey the postulate as restated. The "world-picture" of any observer in my hydrodynamical model is worked out and its astronomical consequences examined. The observed density should show a first-order increase with increasing distance. The evolutionary history of the world on my model is worked out. The notion of "cosmic time" is discussed, and it is shown that events "τ = constant," when the space-time reduces to that of Minkowski, are not simultaneous events in the experience of any observer. The cosmological postulate used by me is stated in terms of experience, and leads directly to formulæ expressing the experience of any observer. The procedure of relativistic cosmology begins with only local experience and with a purely conceptual scheme in a conceptual space-time, and its results require later to be translated into terms of experience.

Oxford:
1933 *June* 25.

ON THE EFFECT OF ERRORS OF OBSERVATION ON THE SPECTROSCOPIC DETERMINATION OF STELLAR ROTATION.

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In recent communications to the Royal Astronomical Society (*M.N.*, **93**, 478, 1933, and *M.N.*, **93**, 508, 1933, referred to below as Paper I and Paper II respectively) a method, developed by the present writer, of determining stellar rotational speeds from observations of spectral line profiles has been described and illustrated. It is clearly desirable to know how far attempts to use the method in practice will be affected by the inevitable errors of observation, and in this note the matter is investigated.

The errors affecting the determination of stellar line profiles (*i.e.* spectrophotometry over a small range of wave-length only) may be classified as follows:—

(1) Errors of observation of the ordinary type, *i.e.* supposing the whole of the procedure and instrumental equipment to be perfect, there will still be the inevitable accidental errors, due for example to the "step" of the measuring instrument. Owing to this type of error any ordinate in the measured profile will be uncertain in magnitude by an amount which may be reduced by repeated observation and averaging.

* Though they apply equally to any relativistic universe.

(2) Systematic errors, of which the principal are—

(2.1) Scattered light in the spectroscope. If this be not completely allowed for, all measured intensities will be altered by a constant amount.

(2.2) Failure of the spectroscope to yield a true spectrum of the light incident on its slit, due to (*a*) finite slit width, and (*b*) finite resolving power. These two effects may in practice be lumped together, and while by no means negligible, should in practice be small, in that, at a conservative estimate, the instrument used should never have less resolving power at the slit width employed than is adequate to reveal absorption lines, say, one-quarter the width of those examined.

(2.3) Photographic effects. These depend on the scale of the image on the photographic plate and upon the type of plate used. In general, errors of this sort are the smaller the shallower and broader the lines investigated. Adequate data are available for the experimenter to dispose his instrumental constants so that these effects may be negligible, but in stellar work the necessity for comparatively short exposures limits freedom of choice, so that such errors may easily be the most important ones, and if the lines are very narrow on the photographic plate (1/10 mm. wide or less) the measures are very suspect on this account.

(2.4) Use of an incorrect origin (*i.e.* λ_m) for $O(\zeta)$. The method used to test for rotational distortion is to compute $\int_{-\infty}^{\infty} O(\zeta) \cos u\zeta d\zeta$ and determine its zeros. In fact $O(\zeta)$, the distorted stellar-line profile, is unknown, and it is $S(\zeta)$, the curve yielded by the spectroscope, that is known, though in some favourable cases in the best work $O(\zeta)$ has been determined from $S(\zeta)$ by applying to the observations corrections known to be applicable to the instruments used.

It is the object of this note to show that this method is independent of errors of observation of type 2.1, 2.2 and 2.4, and to discuss the effect of those of type 2.3 and 1.

First, we consider a function $F(x)$ obtained from a function $f(x)$ by a linear transformation of the closed group of the form

$$F(x) = \int_{-a}^{+a} f(x - \beta t) g(t) dt. \tag{1}$$

Let

$$\left.\begin{aligned} \phi(u) &= \int_{-\infty}^{\infty} f(x) \cos uxdx, \\ \Phi(u) &= \int_{-\infty}^{\infty} F(x) \cos uxdx. \end{aligned}\right\} \tag{2}$$

Then

$$\begin{aligned} \Phi(u) &= \int_{-\infty}^{\infty} \cos ux \int_{-a}^{+a} f(x - \beta t) g(t) dt \\ &= \int_{-a}^{+a} g(t) \int_{-\infty}^{\infty} f(x - \beta t) \cos uxdxdt \\ &= \int_{-a}^{+a} g(t) \int_{-\infty}^{\infty} f(x) \cos u(x + \beta t) dxdt, \end{aligned}$$

which, if $g(t) = g(-t)$, *i.e.* if g is an even function, (3)

$$= \int_{-a}^{+a} g(t) \cos u\beta t dt \times \int_{-\infty}^{\infty} f(x) \cos uxdx$$

$$= G(u\beta) \,.\, \phi(u). \tag{4}$$

Thus the zeros of $\Phi(u)$ are the zeros of $\phi(u)$ together with the zeros of $G(u\beta)$, the effect of the transformation being to introduce no displacements of the zeros of $\phi(u)$ but merely to add to them.

Now with regard to the effect of errors we see that—

(2.1) This cannot affect the zeros, since a constant multiplier only is involved. Instead of $O(\zeta)$ we use $\frac{O(\zeta)+a}{1+a}$, but $\int_{-\infty}^{\infty} \frac{a}{1+a} \cos uxdx = 0$, and so this part is ignorable and $\int_{-\infty}^{\infty} \frac{O(t)}{1+a} \cos utdt$ has the same zeros whatever a be.

(2.2) The effect of finite resolving power is that at any observed point ζ light from all other points $\zeta + t$ $(-\infty < t < +\infty)$ contributes to the image by an amount depending on the optical train and representable by a function $K(t)$, which, for example in the simple case of a perfect grating, is of the form $\frac{\sin^2 nt}{\sin^2 t}$. In practice $K(t)$ is negligible except for a small range of values of t in the neighbourhood of zero, though here we need not assume this, but only that K is an even function. Since we are interested in the estimation of an error of observation, assumed not large compared with the quantity to be found, we may legitimately suppose K can be represented by such an even function. This seems to me a less stringent requirement than supposing K negligible except near the origin, but in practice this is nearly always true. No sensible person tries to measure line profiles without adequate resolving power, when K is a highly concentrated function (see Shane, *L.O.B.*, **16**, No. 449). I do not know of any theoretical or practical instance of an unsymmetrical resolution function K, and if the disposition of the optical train be symmetrical, an appeal to Fermat's principle assures us that K is symmetrical to the second order, but I have in mind the possible effects of lens aberrations at points off the axis. I think some asymmetry might be so introduced, but it is a simple, and as a rule necessary, precaution to work near the axis of the spectrographic camera objective. With regard to the resolution necessary, we shall see presently that it is only required to ensure that the distortion by resolution is less than that due to rotation to avoid trouble.

Thus the relation between the observed profile and the correct one is

$$S(\zeta) = \int_{-\infty}^{\infty} O(\zeta + t)K(t)dt, \tag{5}$$

i.e. the measured profile S is obtained from the true observable profile O by a transformation of type 1 under condition 3, so that the only effect is to introduce new zeros. *The true rotational zeros are left undisturbed*, along with the natural zeros due to $I(\zeta)$.

It may further be observed that when the resolving power of the instrument used is adequate, so that the "resolution" distortion is small, the zeros due to it will only occur for large values of u, so that they will not be mixed with the early rotational zeros. Strictly, we see that if K is an even function the rotational zeros are unaffected, but while theoretically this means that the rotational speed might be determined with any sort of spectrographic equipment, it is necessary to observe that the mixing of the different types of zero which occurs when the resolution is low or the rotational speed small will make it impossible to tell which is which. No doubt it will be possible to identify to some extent the resolving power zeros for any given spectrographic equipment, and so avoid trouble when the distortion due to finite resolving power is only a little less than that due to rotation; but there is the further possibility that K may not be strictly even, so that there is a very slight displacement of the zeros, and I suggest that it is a fair statement to limit the use of the method to the following condition: The minimum rotational speed detectable is twice the velocity of light divided by the resolving power of the spectrograph.

(2.3) Photographic effects in so far as they constitute effectively a loss of resolving power may be similarly treated, but such as the Eberhard effect cannot be so dismissed and constitute a distortion of the observed profile that is not so avoided; it is, moreover, systematic with position in the profile, and unless the observer can claim it as negligible, in the sense that it is less than the "accidental" errors, it will yield misinformation about rotation. A further observation on errors of this type will be made after we have examined the effect of accidental errors.

(2.4) This error is perhaps to be regarded as computational, but it is often not easy to identify the correct origin of ζ in a profile. Fortunately this does not matter; the only effect of an error of this kind is to distort $f(u)$ slightly, but the rotational zeros remain unaffected. Thus if we compute, instead of $\int_{-\infty}^{\infty} O(\zeta)\cos u\zeta d\zeta$, the function $\int_{-\infty}^{\infty} \cos u\zeta O(\zeta+\tau)d\zeta$, this at once transforms to

$$\int_{-\infty}^{\infty} O(\zeta)\cos u(\zeta-\tau)d\zeta$$

$$= \cos u\tau \int_{-\infty}^{\infty} \cos u\zeta O(\zeta)d\zeta + \sin u\tau \int_{-\infty}^{\infty} O(\zeta)\sin u\zeta d\zeta$$

$$= \cos u\tau \int_{-\infty}^{\infty} \cos u\beta tg(t)dt \int_{-\infty}^{\infty} I(\zeta)\cos u\zeta d\zeta$$

$$- \sin u\tau \int_{-\infty}^{\infty} \cos u\beta tg(t)dt \int_{-\infty}^{\infty} I(\zeta)\sin u\zeta d\zeta,$$

so that the rotational zeros $\left(\text{due to } \int_{-\infty}^{\infty} \cos u\beta tg(t)dt\right)$ remain unaltered. Since τ is small the additional zeros will, as a rule, correspond to values of u large enough to be outside the range of u embracing the first few rotational zeros.

The effect of "accidental" errors of type 1 must now be considered. So far as determining rotational speed is concerned these are the most serious errors, except that repeated observation may diminish their importance. We can assess their effect in this manner. Random variations of the ordinates of $O(\zeta)$ will in effect superpose oscillations of irregular form but restricted amplitude on the true form of O. The precise form of the difference between the true and measured curves will depend on the number of observed ordinates and on the amount of smoothing permissible in drawing the curve used through the observed points. If the errors are small they amount then to small ripples of irregular form superposed on the profile, so that we are to expect the effect on the earlier zeros to be small, since in forming the integral for small values of u we average out these ripples largely, and the effect on the higher zeros will be progressively more serious until presently the zeros cease to have any significance owing to this disturbing factor. In other words, we have to ignore zeros corresponding to values of u above a certain limit which will depend on the accuracy of the observations. In the worst cases this limit will exclude the second rotational zero, and we can then only say the measures are useless for our purpose, except perhaps to confirm other determinations ; in the best cases three or even four zeros may be relatively undisturbed.

It is clearly of importance to obtain some measure of the accuracy of any given zero, especially for the purpose of assigning limits of tolerance in accepting zeros as fitting the test for rotation, and judging darkening coefficients. The problem is not one contemplated in the usual theory of errors of observation, but I have attempted to find an analytical expression for the error of a zero determination, and think the rule formulated below is a sufficient criterion in practice.

We start by considering the expression

$$f(u) = \int_{-\infty}^{\infty} O(t) \cos ut \, dt, \tag{6}$$

and remark (1) that $O(t)$ is in fact confined to a range w of t, *i.e.* the width of the observed profile, and (2) that the observed profile is not $O(t)$ but $O(t) + \epsilon(t)$, where $\epsilon(t)$ is the error at the argument t.

For values of u in the neighbourhood of a zero of $f(u)$ we have

$$\int_{-\infty}^{\infty} \{\delta O(t) \cos ut - tO(t) \sin ut \, \delta u\} dt = 0, \tag{7}$$

or

$$\delta u = \frac{\int_{-\infty}^{\infty} \epsilon(t) \cos ut \, dt}{\int_{-\infty}^{\infty} O(t) t \sin ut \, dt}$$

$$= \frac{1}{f'(u)} \int_{-\infty}^{\infty} \epsilon(t) \cos ut \, dt, \tag{8}$$

and in (8) we may effectively replace the limits $-\infty$, ∞ by $-\frac{w}{2}, \frac{w}{2}$.

Now, assuming positive and negative errors of the same magnitude to be equally probable, the average value of (8) is zero ; we require, however, not this information, but something like the average value of the modulus of (8). It is immediately possible to set an upper limit to δu by means of Schwarz's inequality,* obtaining

$$\delta u < \frac{\mu w}{\sqrt{2} f'(u)}, \tag{9}$$

where μ is the mean square error of an ordinate ; but this will as a rule overestimate δu considerably if it be used as an equality, and a closer approximation is needed in practice. This may be obtained as follows, in terms of the actual procedure followed. In computing $f(u)$ from the observations we do not in fact find the integral 6, but by means of some formula of numerical integration we find the algebraic sum of n equidistant ordinates of $O(t) \cos ut$ multiplied by certain factors depending on the numerical integration formula used. I here suppose for simplicity that these factors are all unity. Very little change is caused by giving them their actual values, and we do not want great precision in δu.

Thus we have

$$\delta f = f'(u)\delta u = \sum_{r=1}^{n} \left\{ \epsilon\left(\frac{rw}{n}\right) \cos \frac{urw}{n} \right\} \frac{w}{n}, \tag{10}$$

and we will try to find the mean magnitude of this expression. Now in this formula n represents the number of ordinates used, which for computational purposes may exceed the number of observed ordinates. While this use of a large number of ordinates may be necessary, for arithmetical reasons, to ensure that the computations correctly represent the finally adopted mean smoothed profile, this means that we cannot treat the errors of successive ordinates as independent. Indeed it is not necessarily true that the errors of all the observed ordinates may be regarded as independent, and the number of ordinates that may be measured with errors independent will depend on the experimental procedure—this will be discussed presently; meantime we will take n for our present purpose to mean the number of ordinates that may be regarded as having independent errors.

Using the ordinary relation for the accuracy of the mean, and allowing $\frac{1}{\sqrt{2}}$ for the cosine factor, we have

$$\overline{\delta f} = \frac{\mu w}{\sqrt{2n}}$$

or

$$\overline{\delta u} = \frac{\mu w}{\sqrt{2n} f'(u)}, \tag{11}$$

where μ^2 is the average value of ϵ^2.

This is hardly a rigorous demonstration, but I suggest it as an acceptable estimate ; a more rigorous treatment, by applying Schwarz's inequality to

* See N. Wiener, *The Fourier Integral*, p. 19, Cambridge University Press, 1933.

each interval between ordinates separately and finding the error of the sum, leads to the same result, which should therefore be an estimate on the generous side, and I can add that it is serviceable in practice. The quantity μ is to be obtained as follows. Several independent determinations (*i.e.* from separate negatives) of $O(\zeta)$ are to be averaged into the adopted O, and then n ordinates marked, at the same abscissæ, on the mean curve and on each of the observed curves, and the residuals found for the whole of the ordinates of the observed curves, and from them the mean square error calculated as if they were so many separate observations of a single quantity. We remark in passing here, for later comment, that it may be of importance to notice whether the errors are markedly greater in all curves at some particular abscissa.

It may be observed that δu depends on u, in virtue of the factor $f'(u)$; this has the effect of keeping the percentage error in u roughly constant. A difficulty appears to arise if $f'(u)$ is zero at a zero of $f(u)$, *i.e.* if there is a "touching zero." This difficulty is only apparent, however, and an immediate appeal to the alternative form of (11) giving δf clears up the situation.

The effect of the errors now being considered is to refer $f(u)$ to a false axis of u, displaced from the true one by an amount whose average value we have just calculated. Thus instead of a line for the u axis we should draw a sort of band with the probability concentrated towards the centre, or for convenience in practice we can replace this by two lines parallel to the u axis and each distant from it $\overline{\delta f}$, one above it and one below it. The zeros then may be regarded as coming at any point of $f(u)$ in the regions where the curve lies between the upper and lower of these two lines. The smaller the errors the smaller is $\overline{\delta f}$; thus the closer together are these two lines and the smaller the uncertainty in the position of a zero. The ambiguity arising in the case of a touching zero can be resolved on examining the graph of $f(u)$.

We have now to consider the way in which n, the maximum number of permissible "independent" ordinates, may be determined. Since the only experimental procedure followed at the moment is that of photographic spectrophotometry, it will be sufficient to consider it alone. In other methods it might happen that the correlation between successive ordinates, if sufficiently close together, might be due, say, only to finite instrumental resolving power, but actually the important fluctuations are due to "graininess" of the photographic emulsion and to blemishes of one sort or another in the photographic plate.

It is not easy to specify precisely the interval between ordinates on a photographic plate that is necessary to ensure errors in density measurements being uncorrelated, and further experimental work is clearly necessary. From examination of microphotometer records I am led to the conclusion that there commonly exist in them fluctuations of three types: (1) graininess, (2) minute blemishes giving rise to a waviness in the record of about the same amplitude and frequency as the graininess wriggles, and (3) fluctuations of appreciably greater amplitude but very much lower frequency due

possibly to a gradual variation of emulsion thickness or sensitivity over a region of perhaps $\frac{1}{4}$ mm. in diameter, so that over an appreciable range the record is warped out of its true course, but retaining the same sort of graininess fluctuations as elsewhere. The minute blemishes can be lumped with the graininess and included in the random fluctuations; visible obvious blemishes, *e.g.* pinholes or spots due to air-bells in developing, should be dealt with by rejection of the measures from that region of the plate in which they are found. The third type of error is more insidious in that it is not very concentrated; it is clearly a thing that one may hope to reduce by improvement in experimental technique, and, as it turns out on examination not to be much greater in magnitude than the graininess fluctuations, I suggest the following considerations lead to a fair assessment of the errors in the customary present procedure. In addition to the graininess of the emulsion there is a certain influence of illumination at one point on the density at another point of the plate due to diffusion of light in the image. We can assign a sort of limiting effect of this disturbance when the images of two close lines separated by a clear interval are not distinguished as separate images by the plate—*i.e.* the plate has a finite resolving power, which is closely related to its graininess and also to its speed. The resolving power of a photographic plate depends on the manner of its use and the wave-length of the light illuminating it, but is a fairly precise constant of any given type of plate under fixed conditions. It is defined as the greatest number of lines per millimetre in a grating consisting of alternate dark and clear lines of equal width that are shown when copied by the plate as separate lines; this number varies from about 80 for slow fine-grained process plates to about 40 for fast coarse-grained plates. If our ordinates are not closer together than the lines in the finest reproducible (resolvable) grating, we may then suppose the effect of the light at one ordinate is negligible at the other, and so whatever the reason for the variation between ordinates at the same wave-length on different plates, it is at least not due to the magnitude or the error of neighbouring ordinates. Thus we are led to propose as a suitable value for n the width w of the line on the photographic plate in mm., multiplied by the resolving power of the plate, *i.e.*

$$n = Rw. \tag{12}$$

Of course if the number of measured ordinates is less than this, the number of measured ordinates is the appropriate value of n.

I have been greatly aided in forming some judgment on this matter by the kindness of Professor F. J. M. Stratton, Director of the Solar Physics Observatory, Cambridge, who has supplied me with a large number of microphotometer tracings for examination taken specially with a view to rotational determinations.

With regard to the photographic errors there is one further point of importance, which also concerns any concentrated error whatever its origin.

Professor Sir Arthur Eddington has pointed out to me, in connection with my discussion of accidental error, that the effect of an error concentrated at the centre of the profile is to introduce a displacement of the u axis that

is independent of u, as may be seen from inspection of equation (8). Now, since the slope of $f(u)$ in general changes sign from one zero to the next, the effect on the value of a zero of such a displacement of the axis will be to introduce errors in consecutive zeros of opposite sign, so that the averaging of successive zeros may be expected to diminish errors of this kind. Of course care must be taken that the zeros are consecutive, and that neither is a touching zero in making any such claim; but the possibility exists, and the main interest of it is that the results of the Eberhard effect may be minimised, for this reason, if, as seems to be true, the worst distortion due to the Eberhard effect is at the centre of a profile, so that we can expect some betterment of our results from averaging; the error is not wholly persistent.

To sum up, the method proposed for determining rotational speeds depends on the fact that the shape of a profile distorted by rotation is partly determined by the rotation of the emitting star, and we disentangle this factor from the other factors affecting its shape. The method is strikingly free from errors due to instrumental and other distortions of the profile simply because most of these other distortions are in effect superposed on the rotational distortion but do not alter it, so that unless the rotational distortion is so small compared with the other distortions as to be masked by them, we can still disentangle it from the other factors, and find the rotational speed with no errors other than the ordinary accidental ones common to all physical measurement.

In illustration of these remarks the errors of some of the profiles used in Paper II will now be examined, choosing those for *He* 4387 and *He* 4471 in 44 ι Orionis, for each of which lines two independent profiles were available. This is an instructive case, since the agreement found between the zeros was considered very poor, and the conclusion that the star is rotating, if based only on the evidence of these two lines, very doubtful.

We find for the two *He* 4387 profiles δu is about 10 per cent., *i.e.* this is the error in β, and if this is the main error the value of $10^4\beta$ computed from successive zeros should have an error of about $\pm$ 0·4, and the actual values found were, for $10^4\beta$,

	1st Profile		2nd Profile	
1st zero	3·7		3·9	
		Mean 3·7		Mean 3·7
2nd zero	3·6		3·5	

This is quite satisfactory; the discrepant values come from the two profiles for *He* 4471, which give—

	1st Profile		2nd Profile	
1st zero	4·4		3·4	
		Mean?		Mean 4·0
2nd zero	4·2		4·5	

These profiles differ much in form, and the percentage error in β is here nearly 25, *i.e.* $\pm$1·0 in $10^4\beta$, which leaves all the above figures within the permissible departure from 3·7 but is hardly satisfactory, and clearly either the lines are unsuitable, perhaps through blends, or better determinations

are needed. There is a circumstance that gives a possibility of these results being more concordant than is at first apparent. It turns out that the first zero of the first profile is a double or touching zero, so that if the main displacement of the u axis be due to an error in the profile concentrated at the centre, and if either value be too large the error in the second zero is of the same sign as the error in the first, and so the averaging of these two may not be permissible. This is of course not certain, but may perhaps be regarded as giving some ground for not including these two values in the average of the whole set, but rather only one of them. My point is not that good results can be obtained from poor measurements, or to say that after reconsideration the profiles from this star can be taken as giving a good demonstration of rotational effect, but rather to indicate the way we can use the considerations advanced above to guide the utilisation of results, and to make good determinations more reliable.

The curve for the second profile is of normal type and the residuals are in opposite directions, giving a mean of these two of $10^4\beta = 4{\cdot}0$, and a weighted mean of the results for the *He* 4387 line and *He* 4471 2nd Profile of 3·8; thus the error in the values from the 1st Profile is 0·6 in the first case and 0·4 in the second. If we adopt 0·5 as the mean error we get then 3·8 as the value of $10^4\beta$, which agrees with the 1st Profile, and *not* 4·3 as at first appears the proper value.

I think the case for rotation in this star should be based on *He* 4387, where the errors are small enough for one to feel some confidence in the agreement of the zeros, and, accepting this, *He* 4471 lends support, but could not be used to form a judgment on its own merits.

It is clear that the zeros need interpretation with care, but given the graph of $f(u)$ there is no insuperable difficulty in dealing with fairly large errors of observation.

It is, further, important to notice that, in consequence of the fact that alternate zeros may be affected in the opposite sense by accidental error in the observed profile, it is not in general legitimate to average the resulting values of β without further consideration. The direction of part of the error in any particular rotational zero depends on where it comes in the sequence of zeros of $f(u)$, so that reference to the graph of $f(u)$ may enable one in any particular case to say whether averaging is permissible, or whether other treatment is needed before averaging, as illustrated in the example just given. Since the error δu increases with u (on account of the decrease of $f'(u)$), we presently find δu approaching half the distance between consecutive zeros. It is clearly idle to make use of zeros for values of u of this order or greater, since there is no longer any means of identifying the correct ones. As a rule it will need very good observations to furnish three rotational zeros trustworthily.

It must, further, be remembered that this type of error is not the only one possible, though likely as a rule to be the most important, and until more is known about the proper allowance for darkening and change of line form, departures from equality between values of β from successive zeros may be partly due to this, and if the accidental error is small may exceed it.

The remaining line Mg^+ 4481 used for this star is not very significant; only one profile is available, and the line is very small and shallow compared with the others, so that the errors are likely to be very large, perhaps 50 per cent., and I think the figures from it should be ignored at present, though possibly within experimental error of the true value.

From an examination of the available observational data I think that a profile $O(\zeta)$ obtained by averaging five or more separately determined profiles (I mean from separate individual negatives) should be adequately accurate for testing for stellar rotation, but that in the existing state of spectrophotometry fewer determinations than this will only be sufficient in specially favourable circumstances.

Finally, it is not intended to convey that no effort need be made to remove the errors shown to be unimportant in this paper. They are in any case undesirable, and are only unimportant in so far as determining β is concerned; and even there, while they cannot make the determination incorrect, they may obscure it or make it impossible by the introduction of confusing superfluous zeros. As soon as the problem of computing the original form of the line is tackled the whole of the errors affects the result, and while at the moment this problem may hold less interest than the determination of β, it is clearly worth the effort to solve it; but for satisfactory computations very much greater accuracy in the absolute values of line profiles is needed than appears usual at the moment.

Summary

The effect of accidental and systematic errors of observation in stellar-line profiles is discussed in connection with the method of testing for and determining stellar rotation described in *M.N.*, **93**, 478, 1933.

It is found that the method is independent of distortions of the observed profile due to scattered light and to finite resolving power of the spectrograph employed, in the sense that no numerical corrections to the results are required, though if insufficient resolution be employed the method will fail to yield results owing to the impossibility of identifying the rotational zeros.

A mode of assessing the effect of "accidental" errors is developed, and it is found that special care is necessary in averaging the rotational speeds obtained from different zeros, and in the interpretation of deviations from the mean. It is shown how these difficulties may be avoided.

An estimate is given of the accuracy needed for this process, and the experimental procedure necessary to provide it.

THE CALCIUM IONISATION TEMPERATURE OF THE SUN.

R. v. d. R. Woolley, Ph.D.

§ 1. A well-known thermodynamical formula, first used by Saha, gives the degree of ionisation of an element when it is in an atmosphere at a given temperature and at a given electron pressure. The same formula gives one relation between the temperature and electron pressure in an atmosphere in which a particular element is observed to have a certain degree of ionisation. Information of this kind is afforded by observations of intensities of lines in stellar spectra, namely, the relative intensities of the lines due to a neutral atom and to the ionised atom of the same element, which give one relation between the temperature and electron pressure of the stellar atmosphere.

A difficulty appears at once, since the electron pressure is not constant throughout a stellar atmosphere, but falls away to zero at the surface ; and in the pioneer work on the subject a mean electron pressure for the atmosphere was introduced. This mean pressure was determined from the observations of line intensity (or, which comes to the same thing, from observations of the spectral types at which certain lines appear, reach their maximum, or disappear) by using the Saha relation, and assuming stellar temperatures based on other considerations. During the decade which has elapsed since Saha discovered and applied his formula, a great deal has been found out about the formation of absorption lines, and about their mechanical equilibrium and about the constitution of stellar atmospheres. It is now possible to work in a different way, and predict, from the surface gravity of the star, the electron pressure relevant to the observation of line intensities. We can then use Saha's formula to give an independent value of the stellar temperature. The method is developed in the present paper, and applied to observations of the intensities of the principal lines of neutral and ionised calcium in the Sun.

The outline of the method is as follows. Let k be the coefficient of absorption of continuous radiation in the atmosphere, and let τ be the optical depth at the depth x below the surface of the star : then $\tau = \int_0^x k\rho dx$, where ρ is the density at the depth x in the atmosphere. If k is known, and also g, the surface gravity of the star, then we can calculate P, the electron pressure at the depth τ. We determine the ratio of the number of ionised to neutral atoms by observing the ratio of the squares of the equivalent breadths of the principal lines of the ionised element (at 3933 A.) and neutral element (at 4226 A.) and dividing by appropriate absorption coefficients (oscillator strengths). It is shown in § 3 that we can choose any optical depth —say $\tau = 0{\cdot}3$—and calculate an appropriate factor ϕ so that the actual ratio of ionised to neutral atoms at this particular depth is equal to the ratio found from the observed equivalent breadths multiplied by the factor ϕ. Now

we know the electron pressure at our chosen optical depth, and the observation gives us the ratio of ionised to neutral atoms per c.c. of the atmosphere at this depth. Saha's formula therefore gives the actual temperature at this depth, from which we can calculate the temperature at any optical depth (near the surface) and the effective temperature of the star. All uncertainty as to what depth the observations refer to is eliminated by a single process, the calculation of the factor ϕ.

§ 2. *The Absorption Coefficient for Continuous Radiation*

We shall need to know the value of the coefficient k in order to determine ϕ, and also to determine the electron pressure ϕ. Milne * has made an astrophysical determination of its value. Using the form $k = \alpha P/(RT)^{9/2} = \alpha' P$, he finds the value $\alpha' = 0{\cdot}017$ for the Sun, from the datum that penultimate lines of ionisation potential 7·5 volts attain their maximum width in stars of type Go. More recently, Chandrasekhar † has given a physical theory of the absorption coefficient. His basic formula for a one-constituent atmosphere is

$$k = \alpha P \bar{x}/T^{11/2},$$

where χ is the ionisation potential of the atom in electron-volts.
$\bar{x}$ is the fraction of the atoms ionised.
$\alpha = 5{\cdot}62 \times 10^{19} \times \chi^2/a$,
a being the atomic weight.

Chandrasekhar describes a one-constituent atmosphere which is chosen so as to represent the Sun. The constituent is a fictitious element whose atomic weight is 1·5 and whose ionisation potential is 8·5 volts. It is taken to be 50 per cent. ionised, so that $\bar{x} = \frac{1}{2}$. Then $\alpha = 2{\cdot}58 \times 10^{21}$. Putting this in the form $k = \alpha' P$ we find $\alpha' = 2{\cdot}7$, which differs considerably from Milne's value $\alpha' = 0{\cdot}017$.

The actual composition of the solar atmosphere given by Russell ‡ is shown in Table I.

Table I

Element	*K*	*Na*	*Al*	*Ca*	*Cr*	*Mg*	*Fe*	*Si*	*H*
Atomic weight .	39	23	27	40	52	24	56	28	1
I.P. in volts .	4·33	5·09	5·95	6·09	6·75	7·61	7·63	8·14	13·54
Number of atoms.	50	126	20	40	10	470	126	158	61,500

The mean atomic weight is 1·49.

Now

let m_r be the atomic mass of the rth element,
a_r its atomic weight,
ϵ_r its proportion, by number of atoms per c.c., in the atmosphere,
χ_r its excitation potential in volts,
x_r the fraction of these atoms ionised,
$\bar{m}$ the mean value of m_r,
$\bar{a}$ the mean value of a_r.

* *M.N.*, 89, 37, 1928. † *Ibid.*, 91, 186, 1932. ‡ *Ap. J.*, 75, 337, 1932.

Then

$$k = PT^{-11/2}/\bar{m} \times \sum \epsilon_r m_r a_r x_r = PT^{-11/2}/\bar{a} \times \sum \epsilon_r x_r \chi_r{}^2 \times 5{\cdot}62 \times 10^{19}.$$

Calculating the x_2 on the basis $P = 10^{-4}$ atm. and $T = 6000°$ we get $\sum \epsilon_r x_r \chi_r{}^2 = 0{\cdot}64$ and $k = 0{\cdot}04P$, or $\alpha' = 0{\cdot}04$—a value much closer to Milne's astrophysical estimate than the value $\alpha' = 2{\cdot}7$ found for Chandrasekhar's one-constituent atmosphere. (Note that if $P = 10^{-5}$ atm., $\sum \epsilon_r x_r \chi_r{}^2 = 0{\cdot}86$ and $\alpha' = 0{\cdot}047$, so that the value of α' is not very sensitive to changes in P.)

The mean ionisation is $\bar{x} = 0{\cdot}013$, and the virtual value of α for the atmosphere, defined by $k = \alpha \bar{x} P/T^{11/2}$, is $1{\cdot}84 \times 10^{21}$. The one-constituent atmosphere gives $\alpha = 2{\cdot}58 \times 10^{21}$, so that although the value of α' given by the one-constituent differs widely from that of the mixture, the two values of α are much closer. As can be seen by comparing the values given in the next section of the electron pressure in the mixed atmosphere with those quoted by Chandrasekhar for the one-constituent model, the electron pressure given by the latter is about right, which means that the gas pressure is too small (since $\bar{x} = \frac{1}{2}$ in the model). It is, of course, the abundance of hydrogen, with its high excitation potential, which accounts for the large discrepancy between the mixed atmosphere and the one-constituent model. To conclude this digression, we remark that the main object of this section has been to evaluate k. In what follows we adopt as our formula

$$k = 1{\cdot}84 \times 10^{21} \times \bar{x} P/T^{11/2}.$$

We have already remarked that the computed value of k is insensitive to quite large changes in the adopted mean pressure of the atmosphere. It is also insensitive to changes in the adopted temperature: with $T = 3000°$ and $P = 10^{-5}$ atm., $\alpha = 7{\cdot}5 \times 10^{20}$.

§ 3. *Evaluation of the Factor* ϕ

We now come to the calculation of the factor ϕ which enables us to determine from the observed intensities the ratio of ionised to neutral atoms at a particular optical depth. The actual calculations here given involve some approximation. In the mathematical investigation of the formation of absorption lines we are hindered by the intractability of the differential equations in all except the very simplest cases, where drastic assumptions have been made as to the distribution of the element with optical depth. In fact, solutions are only known in terms of algebraic or tabulated quantities in the special cases η constant * and $1/(1+\eta) = \alpha + \beta\tau$. Milne's method † of investigating stellar surface conditions from observations of line intensities involves the assumption η constant. The present writer ‡ has put forward a method of applying the linear approximation $1/(1+\eta) = \alpha + \beta\tau$. Both

* The coefficient η is defined by $\eta = NMs_\nu/k\rho$, where N is the number of atoms, each of mass M, in the state from which the line is absorbed, per c.c. of the atmosphere, whose density is ρ, and s_ν is the atomic scattering coefficient for the frequency ν.

† Bakerian Lecture, *Phil. Trans.*, A, **228**, 421, 1929.

‡ *M.N.*, **92**, 482, 1932.

these methods involve the measurement of the width of a line at some particular ordinate r.*

The method given in the present section makes use of the same linear approximation $1/(1+\eta) = \alpha + \beta\tau$, and differs from the writer's earlier method only in that the observed quantity is made the equivalent breadth of the line, $W = \int(1-r)d\lambda$. In this way we escape from troublesome corrections due to the finite resolving power of the spectrograph used in the observations.

We start by making as good as possible an estimate of the distribution of Ca and Ca^+ with optical depth, assuming a definite value of T_e, the effective temperature of the star. We then choose a set of values of η (the set actually chosen is 10, 3, 1, 0·3 and 0·1) at some definite optical depth ($\tau = 0{\cdot}3$).

Since
$$\eta = \frac{NMs_\nu}{\kappa\rho} \quad \text{and} \quad s_\nu = e^2/4\pi mc\,\frac{f\omega}{(\nu-\nu_0)^2},$$

where f is the oscillator strength of the line and ω is the natural breadth, these values of η correspond to a set of values of $(\nu-\nu_0)$ (in the ratio $1/\sqrt{10} : 1/\sqrt{3} : 1 : 1/\sqrt{0{\cdot}3} : 1/\sqrt{0{\cdot}1}$). To each value of η at $\tau = 0{\cdot}3$ there corresponds a set of values of η at all other optical depths, $\nu-\nu_0$ being fixed, and in each case we plot $1/(1+\eta)$ against τ and choose α and β to give the best fit in the linear approximation $1/(1+\eta) = \alpha + \beta\tau$. The blackening r for this value of $\nu-\nu_0$ is given† by

$$r = \frac{\alpha + \sqrt{(\tfrac{1}{3}\alpha)}\,(2+\beta)\kappa_1/\kappa_0}{1 + \sqrt{(\tfrac{1}{3}\alpha)}\,2\kappa_1/\kappa_0}, \qquad 3.1$$

the argument of the Bessel functions $\kappa_0(x)$ and $\kappa_1(x)$ being $\sqrt{\frac{12\alpha}{\beta^2}}$. When the blackening r is calculated for a set of values of $(\nu-\nu_0)$ the theoretical contour of the line can be drawn, using an arbitrary value of $Nf\omega\lambda^4/k_\nu$.‡ We draw these contours for the neutral and for the ionised line, choosing $Nf\omega\lambda^4/k_\nu$ the same for each line. Let the equivalent breadths computed in this way be W_0' for the neutral line and W_1' for the ionised line. The actual equivalent breadth W_0 of the neutral line will differ from the fictitious contour because n_0, the actual number of neutral atoms per c.c. at $\tau = 0{\cdot}3$,

* That is to say, at a point where the intensity of the light within the line has fallen to the fraction r of the intensity of the continuous background just outside the line.

† Eddington, *M.N.*, **89**, 620, 1929.

‡ Since
$$s_\nu = e^2/4\pi mc\,\frac{f\omega}{(\nu-\nu_0)^2} = \frac{\lambda^4}{(\lambda-\lambda_0)^2}\,\frac{f\omega e^2}{4\pi mc^5},$$
$$\eta = \frac{NMs_\nu}{k\rho} = \frac{Nf\omega\lambda^4}{\kappa_\nu(\lambda-\lambda_0)^2}\,\frac{Me^2}{4\pi mc^5\rho}.$$

For a given η (and hence a given value of r) $(\lambda-\lambda_0)^2 \propto Nf\omega\lambda^4/k_\nu$.

Since $W = \int(1-r)d\lambda = 2\int_0^1(\lambda-\lambda_0)dr$, $W^2 \propto Nf\omega\lambda^4/k_\nu$ for a given distribution of η with τ.

is not equal to N_0, so that $W_0 = W_0'\sqrt{n_0/N_0}$. Similarly $W_1 = W_1'\sqrt{n_1/N_1}$. Now $N_0 f_0 \omega_0 \lambda_0^4/k_0 = N_1 f_1 \omega_1 \lambda_1^4/k_1$, so that

$$\frac{n_1}{n_0} = \left(\frac{W_1}{W_0}\right)^2 \times \left(\frac{W_0'}{W_1'}\right)^2 \times \frac{\lambda_0^4 f_0 \omega_0 k_1}{\lambda_1^4 f_1 \omega_1 k_0} = \left(\frac{W_1}{W_0}\right)^2 \times \frac{\lambda_0^4 \omega_0 f_0 k_1}{\lambda_1^4 \omega_1 f_1 k_0} \times \phi, \qquad (3.2)$$

where ϕ is the correcting factor, equal to the calculated value of $(W_0'/W_1')^2$. This is to say that n_1/n_0, the ratio of the number of ionised atoms to the number of neutral atoms per c.c. at the prescribed depth $\tau = 0{\cdot}3$, is found in the ordinary way from the observed equivalent breadths and appropriate absorption coefficients, and multiplying by the factor ϕ. The procedure is rigorously correct, but for (1) the introduction of the approximation $1/(1+\eta) = \alpha + \beta\tau$, and (2) uncertainty in the distribution of *Ca* and Ca^+. Happily the factor ϕ for $\tau = 0{\cdot}3$ is not sensitive even to large changes in assumed distribution. The variation of k with ν has been given by Pannekoek (*M.N.*, **91**, 139, 1930. See especially his formula (10) on p. 152). This formula gives for $T = 6000°$, $k_{3933}/k_{4227} = 1{\cdot}35$. Let T be the effective temperature of integrated radiation at the depth τ. Then, as is shown in Eddington's *Internal Constitution of the Stars*, p. 323, $T^4 = T_0^4(1 + \frac{3}{2}\tau)$, and the effective temperature of the star T_e is given by $T_e^4 = 2T_0^4$. If the matter at the depth τ was in thermodynamical equilibrium with the radiation there at a temperature T, the ionisation of calcium (potential $I = 6{\cdot}09$ volts) would be given by

$$\log \frac{x}{1-x} = -\frac{5040I}{T} + \tfrac{5}{2}\log T - 6{\cdot}5 - \log P + \log\left(\frac{2B'(\tau)}{B(\tau)}\right), \qquad (3.3)$$

where x is the fraction of the calcium atoms in the ionised state, and $B'(\tau)$, $B(\tau)$ are the partition functions for ionised and neutral calcium. They are analogous to statistical weights in the case of the distribution of atoms of the same kind among various states of excitation. The number 2 is the symmetry number of the electron. Unless the temperature is high enough to give weight to higher states, $B'(\tau)$ and $B(\tau)$ are the statistical weights of the ground states of the ionised and neutral atom. The ground state of Ca^+ is $4^2S_{1/2}$, for which $q = 2$, and for *Ca* the ground state is 4^1S_0, with $q = 1$. The latter must be doubled, since there are two valence electrons, which leaves us with $B'(\tau) = B(\tau)$. At temperatures of 6000° this relation is not disturbed by the higher states.

According to Pannekoek * there is a departure from thermodynamical equilibrium arising from the fact that the radiation is flowing in one direction only at the surface. Judged by the average flow in both directions, outflow and (zero) inflow, the temperature is T_0; but the frequency distribution of the outflowing radiation, and its amount, correspond to $T_e = 2^{1/4}T_0$. The ionisation at the surface is given by

$$\log \frac{x}{1-x} = -\frac{5040I}{T_e} + \tfrac{5}{2}\log T_e - 6{\cdot}5 - \log P + \tfrac{3}{2}\log \frac{T_0}{T_e} + \log\left(\frac{B'(\tau)}{B(\tau)}\right). \quad (3.31)$$

* *B.A.N.*, 3, 207, 1926.

Both T_0 and T_e appear in the equation, because the ionisation depends partly upon the velocity of the free electrons, which is controlled by T_0, and partly upon the radiation, controlled by T_e. This equation requires extension before it can be applied to the interior, say at $\tau = 0{\cdot}3$. Suppose that the radiation $\mathcal{J}(\theta)$ in the direction θ in the interior of the atmosphere has the value $\mathcal{J}_1$ for outflowing radiation $0 < \theta < r/2$, and the value $\mathcal{J}_2$ for inflowing radiation $r/2 < \theta < \pi$. Then (Eddington, *Internal Constitution of the Stars, loc. cit.*)

$$\mathcal{J} = \tfrac{1}{2}(\mathcal{J}_1 + \mathcal{J}_2), \qquad H = \tfrac{1}{4}(\mathcal{J}_1 - \mathcal{J}_2), \qquad \mathcal{J} = H(2 + 3\tau),$$

giving $\mathcal{J}_1 = 4H(1 + \frac{3}{4}\tau)$ and $\mathcal{J}_2 = 3H\tau$. At the surface, $\mathcal{J}_2 = 0$ and $\mathcal{J}_1 = 4H$. Now T_0 is the temperature of the average radiation $\mathcal{J}$, and T_1 is that of the radiation $\mathcal{J}_1$, so that $acT_0^4 = 2H$ and $acT_1^4 = 4H$. We get $T_1 = 2^{1/4}T_0 = T_e$, giving Pannekoek's result (3.31) for the surface $\tau = 0$.

At any other optical depth τ, $\mathcal{J}_1 = 4H(1 + \frac{3}{4}\tau)$, giving $acT_1^4 = 4H(1 + \frac{3}{4}\tau)$. For the reverse radiation $\mathcal{J}_2 = 3H\tau$, but the temperature of this radiation is not given by its amount. (This can be seen by considering the limit $\tau \to 0$; clearly the reflected radiation does not correspond in frequency distribution to an infinitely low temperature.) The temperature of the reverse radiation $\mathcal{J}_2$ cannot differ greatly from T, the local temperature. The influence of $\mathcal{J}_2$ on the ionisation is in any case not very great at small optical depths: it may amount to 10 per cent. at $\tau = \frac{1}{2}$: so that we fix its temperature at T for convenience in the mathematical treatment.

Following Pannekoek (*loc. cit.*), and writing $\left(\frac{Px}{1-x}\right)_T$ for the value of $\frac{Px}{1-x}$ in thermodynamical equilibrium at an electron pressure P and temperature T, Pannekoek shows that

$$\frac{Px}{1-x} = \left(\frac{Px}{1-x}\right)_T \times \frac{\int_{\nu_0}^{\infty} \nu^{-3} I_\nu d\nu}{\int_{\nu_0}^{\infty} \frac{8\pi h}{c^2} e^{-h\nu/RT} d\nu},$$

ν_0 being the lower limit for the frequency of light capable of ionising the atom.* Introducing the temperature for the two streams of radiation we find

$$\frac{Px}{1-x} = \left(\frac{Px}{1-x}\right)_T \times \tfrac{1}{2}\left\{\frac{T_1}{T}\exp\left\{-\frac{I}{R}\left(\frac{1}{T_1} - \frac{1}{T}\right)\right\} + \frac{3\tau}{2+3\tau}\right\}. \qquad (3{\cdot}32)$$

The ionisation for a given τ, T and P can now be calculated from (3·3) and (3.32).

* This limit is taken as the limit of the frequency capable of ionising the atom in the ground state. Longer wave-lengths will be capable of ionising the atom when it is in excited states. Our analysis is correct if the number of atoms in the excited states is controlled by the radiation temperature T_1 and not by the local (collision) temperature T_0.

Neglecting the radiation pressure, we have as the equation of equilibrium for the gas pressure p at a depth x below the surface,

$$dp = g\rho dx.$$

Multiplying by k,

$$kdp = gd\tau. \tag{3.4}$$

Now

$$k = \alpha P\bar{x}/T^{11/2}$$

and

$$P = \frac{\bar{x}}{1+\bar{x}}p.$$

We neglect the variation of $\bar{x}$ with τ. (This approximation is also used by Chandrasekhar.) Then

$$dP = \bar{x}/1+\bar{x}\,.\,dp.$$

Substituting for k, (3.4) gives

$$(1+\bar{x})\,.\,\alpha\,.\,PdP = gT_0^{11/2}(1+\tfrac{3}{2}\tau)^{11/8}d\tau.$$

Integrating, we get

$$\tfrac{1}{2}\alpha(1+\bar{x})P^2 = gT_0^{11/2}\int_0^\tau (1+\tfrac{3}{2}\tau)^{11/8}d\tau.$$

Finally, neglecting $\bar{x}$ compared with 1,

$$P^2 = \frac{2g}{\alpha}T_0^{11/2}\int_0^\tau (1+\tfrac{3}{2}\tau)^{11/8}d\tau, \tag{3.41}$$

which gives P at the depth τ when g and T_0 are known. The ionisation of calcium is then found from (3.3) and (3.32).

We now suppose that the atmosphere is well mixed, *i.e.* that if $n = n_0 + n_1$, n/ρ is constant. The variation of η with τ is now given by

$$\eta = \frac{nMxs_\nu}{k\rho} \propto \frac{x(1+\tfrac{3}{2}\tau)^{11/8}}{P} \tag{3.51}$$

for the neutral atom,

$$\eta = \frac{nM(1-x)s_\nu}{k\rho} \propto \frac{(1-x)(1+\tfrac{3}{2}\tau)^{11/8}}{P} \tag{3.52}$$

for the ionised atom.

As has been said before, we choose a value of η, say $\eta = 10$, at the depth $\tau = 0{\cdot}3$. The values of η at all other optical depths are then fixed by (3.51) or (3.52) as the case may be. We plot $1/(1+\eta)$ against τ and choose α and β to get the best fit to the approximation $1/(1+\eta) = \alpha + \beta\tau$, and calculate the blackening r from α and β by the formula (3.1). Details of the calculation for the Sun ($T_e = 6000$, $g = 2{\cdot}74\,.\,10^4$) are shown in Table II, which shows the calculations of x and $(1-x)$, and of η as a function of τ: the tabular value of η contains an arbitrary factor depending on the value of $\nu - \nu_0$ chosen.

TABLE II

	0·6	0·5	0·4	0·3	0·2	0·1	0·05
T	5922	5802	5674	5536	5386	5224	5137
$10^5 P$, in atmospheres	8·35	7·33	6·31	5·24	4·10	2·76	1·91
$\left(\frac{x}{1-x}\right)_T$ from (3.3)	135	114	95	78	66	60	66
Factor from (3.32)	2·09	2·32	2·58	2·89	3·48	4·16	4·62
$\frac{x}{1-x}$ from (3.32)	282	264	246	226	230	250	306
x	·0035	·0038	·0040	·0044	·0043	·0040	·0033
$1-x$	·996	·996	·996	·996	·996	·996	·997
η, neutral atom	73	80	87	100	107	125	135
η, ionised atom	91	93	95	100	109	138	181

Details of the calculation of r are shown in Table III. In the case of the ionised atom it is not possible to represent $1/(1+\eta)$ as a linear function of τ throughout the region $0<\tau<0{\cdot}6$, and the behaviour of $1/(1+\eta)$ is represented by two straight lines. An example of this procedure, and appropriate formulæ, have been given by the writer in *M.N.*, **92**, 482, 1932.

TABLE III

Value of η at $\tau=0{\cdot}3$	Neutral Line		Ionised Line	
	Approximation for $\frac{1}{1+\eta}$	Value of r	Approximations for $\frac{1}{1+\eta}$	Value of r
10	$0{\cdot}06+0{\cdot}10\tau$	0·27	$0{\cdot}07+0{\cdot}06\tau$; $0{\cdot}02+0{\cdot}5\tau$	0·20
3	$0{\cdot}17+0{\cdot}26\tau$	0·50	$0{\cdot}21+0{\cdot}12\tau$; $0{\cdot}06+1{\cdot}4\tau$	0·43
1	$0{\cdot}43+0{\cdot}26\tau$	0·74	$0{\cdot}44+0{\cdot}18\tau$; $0{\cdot}25+1{\cdot}5\tau$	0·65
0·3	$0{\cdot}72+0{\cdot}16\tau$	0·90	$0{\cdot}73+0{\cdot}12\tau$; $0{\cdot}56+1{\cdot}4\tau$	0·86
0·1	$0{\cdot}90+0{\cdot}05\tau$	0·97	$0{\cdot}89+0{\cdot}06\tau$; $0{\cdot}78+1{\cdot}0\tau$	0·96
Equivalent breadth in arbitrary units	$W=153$		$W=199$	
Value of $\phi=(\frac{153}{199})^2=0{\cdot}59$				

§ 4. *The Line-absorption Coefficients*

We have already mentioned that the absorption coefficient for the frequency ν within a line whose central frequency ν_0 is equal to

$$e^2/4\pi mc \times \frac{f\omega}{(\nu-\nu_0)^2}.$$

This formula does not apply to the centre of the line, $\nu = \nu_0$, but is applicable to all frequencies used in practice. Here f is the oscillator strength, and ω is the damping constant of the line. Unfortunately for the general progress of astrophysical spectroscopy, it is usually impossible to give a numerical value to the quantity f for each line in the present state of our knowledge of theoretical physics. It is possible that astrophysical observations may themselves best determine these quantities. But some results can be obtained from the theoretical physics of the present day, and, in particular, we can deduce the constants for the calcium lines that are the subject of the present discussion. An account of the method has been given by Minnaert and Mulders.*

The oscillator strengths f are found from the sum rule (*Summensatz*), which states that the sum of the oscillator strengths of all the transitions connected with a state, including transitions resulting in continuous radiation, is equal to the number of valence electrons of the element or ion in question, *e.g.* 2 for *Ca* and 1 for Ca^+. Writing f_{abs} for the f values of absorption lines, *i.e.* lines for which the state considered is the lower state, and f_{em} for emission lines, for which the state considered is the upper state, the sum rule gives

$$\sum f_{\text{abs}} + \sum f_{\text{em}} = s, \qquad s = 1, 2, \ldots \qquad 4.1$$

If we know f we can calculate ω. In the classical theory, ω is equal to $\frac{8\pi^2 e^2 {\nu_0}^2}{3mc^3} = \omega_{kl}$, but according to Weisskopf and Wigner † ω is equal to the sum of the spontaneous Einstein coefficients for the upper and lower state of the line. These are related to the f's by the equations

$$f_{\text{abs}} = \tfrac{1}{3}\frac{q_x}{q_n}\frac{A_{xn}}{\omega_{kl}}, \qquad 4.21$$

$$f_{\text{em}} = -\tfrac{1}{3}\frac{A_{ny}}{\omega_{kl}}, \qquad 4.22$$

where x is any state above the state n considered, and y is any state below n. q_x, q_n are statistical weights.

Weisskopf and Wigner give

$$\omega = \sum_x A_{xn} + \sum_y \mathrm{A}_{ny}, \qquad 4.3$$

the sum being taken over both upper and lower states.

Consider first the *Ca* line at 4227, which is $4^1S_0 - 4^1P_1^0$. Here $q = 1$ for the lower state, and $q = 3$ for the upper state. We neglect all other transitions from the 4^1S_0 state, and apply (4.1). This gives $(f_{\text{abs}})_{4227} = 2$. Hence from (4.21) $A_{4227} = \frac{1}{3} \cdot \frac{3}{1} \cdot (f_{\text{abs}})_{4227} \times \omega_{kl} = 2\omega_{kl}$. There are no spontaneous transitions from the lower state $^1S_0$, which is the ground state of *Ca*. The only spontaneous transition from $4^1P_1^0$ is the line 4227 itself. Accordingly (4.3) gives $\omega = A_{4227} = 2\omega_{kl}$. Now consider the lines Ca^+, *H*

* *Zs. f. Astrophysik*, 2, 165, 1931.
† *Zs. f. Phys.*, 63, 54, 1930.

and K, $4^2S_{1/2} - 4^2P^0_{3/2,\ 1/2}$. Part of the Grotrian diagram is shown in fig. 1. Neglecting all transitions from the ground state except H and K we find

$$f_H + f_K = 1,$$

so that $f_H = \frac{1}{3}$, $f_K = \frac{2}{3}$ by the ordinary multiplet rules.

Applying (4.21) we find $A_H = \omega_{kl}$ and $A_K = \omega_{kl}$. We would get $\omega = \omega_{kl}$ for both lines but for the presence of the spontaneous transitions

$$4^2P^0_{3/2,\ 1/2} \rightarrow 3^2D_{3/2,\ 5/2}.$$

According to Zwaan * the probability of the spontaneous transition $4^2P^0_{1/2} \rightarrow 3^2D_{3/2}$ is 0·09 of the probability of the transition $4^2P^0_{1/2} \rightarrow 4^2S_{1/2}$.

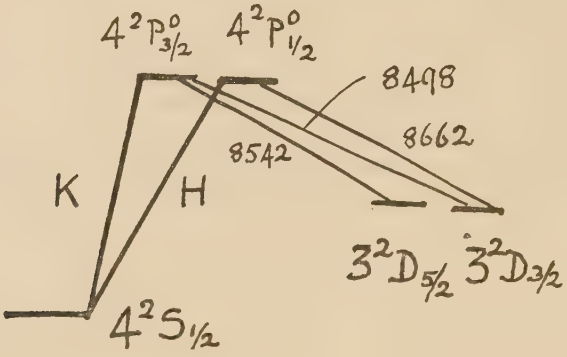

Fig. 1.—Ca^+.

Hence for both H and K, $\omega = 1{\cdot}09\omega_{kl}$. It should be remembered that ω_{kl} depends in each case on the frequency of the line concerned.

§ 5. *Observations. The Solar Temperature*

Contours of the K line at 3934 A. in the solar spectrum have been published by Schwarzschild, Unsöld and Minnaert.† The last-named observer found a substantial difference between his own contour and that given by Schwarzschild, and concluded that his result was in error, because he had not observed the continuous background sufficiently far out from the line. He accordingly applied a correction. In view of the uncertainty, the present writer made an independent determination, which gives a value very close to Minnaert's uncorrected result. The various results are shown in Table IV. We must not overlook the possibility of real changes in the contour of K from time to time.

Table IV

Equivalent Breadth of Ca^+ 3934 K, *in Angstroms*

Unsöld	12·6
Minnaert, uncorrected	9·6
Woolley	10·3

* *Die Naturwissenschaften*, **17**, 121, 1929.

† Minnaert, *Zs. f. Phys.*, 34, 614, 1927; Unsöld, *ibid.*, 44, 793, and **46**, 765, 1927.

The instrument used by the present writer is the McClean solar spectrograph at the Solar Physics Observatory, Cambridge. The instrument is of the Littrow type, and has previously been used with a plane grating. The grating has now been replaced by a set of prisms of excellent quality made by Hilger.

Normally three 60° prisms and one 30° reflecting prism are used, giving a dispersion of 0·62 A. per mm. at 4200 A., the focal length being 14 feet. The aperture of the prisms is $2\frac{1}{4}$ inches.

To observe *H* and *K*, whose wings extend considerably into the neighbouring spectrum, two of the 60° prisms were removed, and the resolving power sacrificed in order to get a large range of spectrum on the quarter-plate used. The lines are, of course, so wide that no particularly great resolving power is required to measure their equivalent breadth. The full set of prisms was used in the determination of the equivalent breadth of *Ca* 4227. The photometric technique has already been described.*

In addition to the Cambridge measurements, 4227 has been repeatedly measured on plates exposed in Mr. Evershed's spectrograph at Ewhurst. There is a small systematic difference between the two instruments. The results are tabulated in Table V.

TABLE V

Equivalent Breadth of 4227 Ca

Mean of 5 plates taken at Cambridge	$W = 1194 \pm 56$ milliangstroms
,, 3 ,, ,, Ewhurst	$W = 1105 \pm 36$,,

The adopted value for the equivalent breadth of *Ca* 4227 is 1·194 angstroms, and that for Ca^+ 3933 is 10·3 angstroms. These values, which are the mean values for observations made in 1933 April at Cambridge with the McClean prism spectrograph, differ slightly from general mean values taking into account other observers and other instruments; but the ratio is best determined from measurements made at the same time and with the same instrument (although there was a different dispersing train in the two cases). We get accordingly, for the ratio of the number of ionised calcium atoms per c.c. to the number of neutral atoms per c.c. at the depth $\tau = 0{\cdot}3$, from equation (3.2),

$$n_1/n_0 = \left(\frac{W_1}{W_0}\right)^2 \times \frac{\lambda_0^4 f_0 \omega_0 k_1}{\lambda_1^4 f_1 \omega_1 k_0} \times \phi$$

$$= \left(\frac{10{\cdot}3}{1{\cdot}194}\right)^2 \times \frac{2 \times 2}{\frac{2}{3} \times 1{\cdot}09} \times \left(\frac{4227}{3933}\right)^2 \times 0{\cdot}59 \times 1{\cdot}35$$

$$= 377.$$

The electron pressure at this depth is known from (3.41) if T_0 is known. The easiest way to find the appropriate value of T_0 is by interpolation from a few trial values, such as are shown in Table VI.

* *Annals of the Solar Physics Observatory*, **3**, pt. 2; *M.N.*, **92**, 806, 1932.

Table VI

Values of n_1/n_0 *at* $\tau = 0{\cdot}3$ *for Various Values of* T_0

T_0	5046	5130	5210	5290	5380
P from (3.41) in atm. $\times 10^{-5}$	5·24	5·48	5·75	6·01	6·27
n_1/n_0 at $\tau = 0{\cdot}3$ from (3.3) and (3.32)	226	270	317	367	440
$T_e = 2^{1/4} T_0$	6000	6100	6200	6300	6400

From Table VI it appears that the observations give $T_0 = 5300$ and $T_e = 6310$. The effective temperature of the centre of the disc is $(\frac{5}{4})^{1/4} T_e$ or 6690°.

§ 6. *Sources of Error, and the Probable Error of the Determination*

The sources of error in the determination of the solar temperature that we have given are of three kinds :

(*a*) The ionisation in the solar atmosphere may differ from that given by (3.32).

(*b*) On account of the approximations involved there may be an error in the calculated value of the factor ϕ.

(*c*) In addition, there will be the probable errors in the adopted values of the equivalent breadths of the calcium lines.

Taking (*a*) first, further research will, no doubt, throw light on this question. Meanwhile, the agreement between the temperature found in this way and those found by other methods show that our extension of Pannekoek's modification represents an advance over the crude application of Saha's formula, equation (3.3) alone, which gives $\phi = 0{\cdot}67$ and $T_e = 6800°$. Similarly there is reason to have some degree of confidence in the adopted value of the absorption coefficient, which is an adaptation of Chandrasekhar's value.

(*b*) A little experience in calculating r from formulæ of the type $1/(1+\eta) = \alpha + \beta\tau$ will show that the values of r are not very sensitive to slight changes in α and β. (See, for example, an earlier paper by the writer, *M.N.*, **92**, 482, 1932.) The values of r are probably correct to within 1 or 2 per cent., and it is unlikely that ϕ is in error by as much as 5 per cent.

(*c*) Although there is usually some discrepancy between observed equivalent breadths of the same line determined with different instruments, a set of determinations on a single line with the same instrument usually shows a probable error of about 5 per cent. Presumably the probable error of the ratio of two equivalent breadths determined by the same instrument is about the same, and although it is a wide extrapolation of this principle to suppose that the systematic errors will be roughly the same for a line of equivalent breadth 1 A. measured with 3 prisms as for a line of equivalent breadth 10 A. measured with 1 prism, yet it seems improbable that the errors due to (*b*) and (*c*) taken together will exceed 10 per cent. The corresponding error in T_e is about 50°.

Our final result is therefore

$T_0 = 5300°$.
$T_e = 6310° \pm 50°$ for the sunlight integrated over the disc.
$T_e = 6690°$ for the centre of the disc.

Table VII shows a collection of values of T_e for sunlight integrated over the disc given by a number of authorities and found by a variety of methods.

TABLE VII

Authority	Method	T_e for the Integrated Solar Disc	Reference
Milne	Stefan-Boltzmann Law	5740	*M.N.*, **81**, 361, 375, 1921. *Phil. Trans.*, **223**, 201, 1922.
Bernheimer	Wien's Law. Mean value for a number of determinations.	6079	*Handbuch der Astrophysik*, **4**, 50.
Fabry and Buisson	Planck's Law, 2932–3940 A.	5830–6000	*C.R.*, **175**, 156, 1922
Abbot and others.	Planck's Law, 5000–10,000 A.	6000	*Smithsonian Misc. Coll.*, **74**, 1923.
Lindblad	Planck's Law	5950	*Nova Acta Upsal*, **4**, 1923.
Brill	Planck's Law—Bolometric	5775	Bernheimer, *Handbuch der Astrophysik*, **4**, 52.
	Visual	6075	
	Photographic	5835	
Plaskett	Planck's Law, 4800–6700 A.	6300–6600	*Pub. D.A.O.*, **2**, 213, 1923.
Present work	Ionisation of calcium	6310	

§ 7. *Astrophysical Determination of the Oscillator Strengths of* Ca *Lines*

Minnaert has published an example of an astrophysical determination of a spectroscopic constant, namely, the damping constant ω of the *Mg* triplet lines. As he says, it is a very instructive calculation. The present writer has made a set of measurements of the equivalent breadths of a set of *Ca* lines which leads to a determination of their oscillator strengths, though not by the method that Minnaert * used. As no account of the new quantum theory of the widths of absorption lines has yet appeared in English in a journal accessible to astrophysicists, the theoretical work will be shown here in some detail.

Fig. 2 shows part of the Grotrian diagram for *Ca*. The lines measured were the principal line $4^1S_0 - 4^1P_1^0$ at 4227 A., and the lines which originate, in absorption, in the metastable 4^3P^0 state, the triplet $4^3P^0 - 5^3S$ in the red (6162, 6122 and 6102 A.), and the $4^3P^0 - 4^3D$ multiplet in the blue (4456 – 4425 A.). Unfortunately, the multiplet $4^3P^0 - 3^3D$ is so far in the remote infra-red that it could not be measured with the apparatus available. Although there are three absorption coefficients—damping constant, oscillator strength and Einstein transition probability—there is only one funda-

* Minnaert and Mulders, *Zs. f. Astrophysik*, **2**, 176, 1931.

mental constant to be determined for each line: the other two constants follow from it. We may of course take any one constant as fundamental and then define the others in terms of it. It seems most agreeable to take the Einstein transition probability A of the line as fundamental. Let state 2 be the upper state and state 1 the lower state of the line, and let there

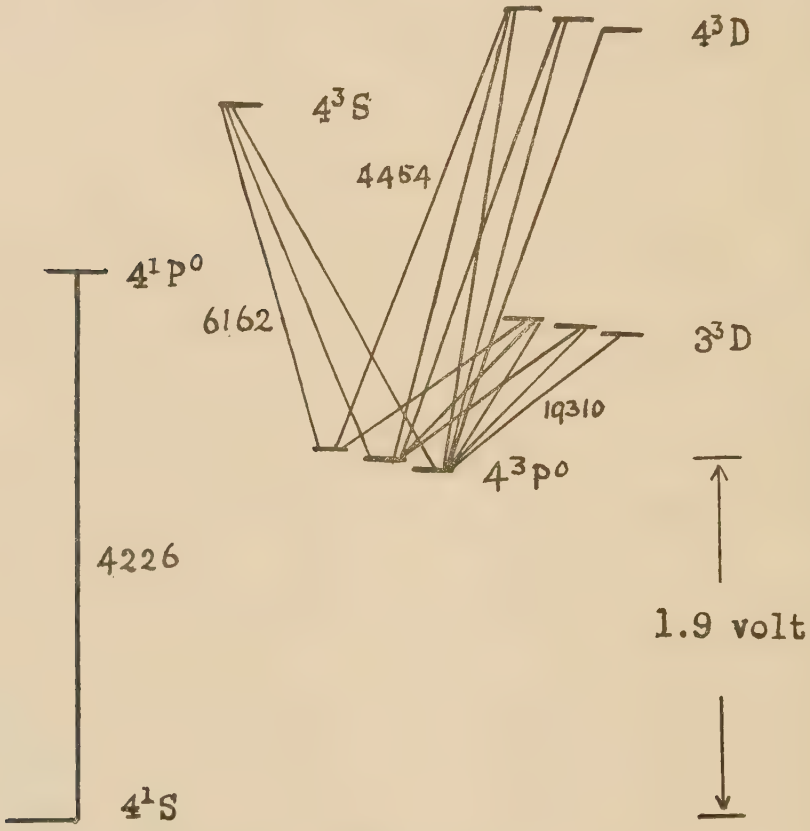

FIG. 2.—*Ca.*

be n_2 atoms in the upper state 2. Then the number of spontaneous transitions $2 \rightarrow 1$ is $n_2 A$ per second. This defines A.

Now let q_2, q_1 be the statistical weights of states 2 and 1 respectively.

ω_{kl} be the "classical damping constant of the line," $= \dfrac{8\pi^2 e^2 \nu^2}{3mc^2}$.

s the number of valence electrons, equal to 2 for neutral *Ca*.

Then the oscillator strengths of the line, one for emission and one for absorption, are defined by

$$f_{\text{abs}} = \tfrac{1}{3} \frac{q_2}{q_1} \frac{A}{\omega_{kl}},$$

$$f_{\text{em}} = -\tfrac{1}{3} \frac{A}{\omega_{kl}}.$$

The importance of these oscillator strengths lies in the sum rule (*Summensatz*), which states that for any sub-level the sum of the oscillator strengths of all the lines connected with it is equal to the number of valence electrons, or

$$\sum f_{\text{abs}} + \sum f_{\text{em}} = s.$$

The sum rule gives the absolute values of a set of A's if their relative values are known. Within a multiplet, the relative values of the A's are given by the ordinary multiplet intensity rules, the multiplet intensity being proportional to $q_2 A$. In the special case where there is only one multiplet transition from states 1 to states 2 the f's and A's can be evaluated, but in the case of the *Ca* lines there are three multiplets arising from the 4^3P^0 states. To find the A's we must know not only the relative A's within each multiplet, but the relative values from one multiplet to another. It is at this point that the astrophysical observation of relative intensity is used. Supposing, however, that the A's are known, the damping factor ω is given by the sum of the A's of all the lines which are spontaneous transitions from state 2 and from state 1, or $\omega = \sum_2 A + \sum_1 A$.

The astrophysical importance of these coefficients lies in the fact that the atomic scattering coefficient s_ν can be found from them, since

$$s_\nu = e^2/4\pi mc \times \frac{f_{\text{abs}}\,\omega}{(\nu - \nu_0)^2}.$$

Two results which are helpful in carrying out actual calculations may be noted. The constant ω is the same for all members of a multiplet, and the sum of the A's in any multiplet must be the same for each (upper) sub-level.

We turn now to the astrophysical question of the equivalent breadths of the lines. If two absorption lines arise from the same lower sub-level, the ratio of the squares of their equivalent breadths are proportional to the values of $\lambda^4 f\omega/k$ for the two lines.

Now consider the simplified case illustrated by fig. 3, which represents a singlet system in which the state P is metastable. There are no other spontaneous transitions from the S and D states. We take $s = 2$. Using the subscript α for the $^1P_1 - {}^1S_0$ line and β for the $^1P_1 - {}^1D_2$ line, we can evaluate the f's and ω's in terms of A_α, A_β as follows :—

$$f_{\alpha,\,\text{abs}} = \tfrac{1}{9}\frac{A_\alpha}{\omega_{kl,\,\alpha}}, \qquad f_{\beta,\,\text{abs}} = \tfrac{5}{9}\frac{A_\beta}{\omega_{kl,\,\beta}}.$$

$$\omega_\alpha = A_\alpha, \qquad \omega_\beta = A_\beta.$$

Applying the sum rule to the level $^1P_1$, we get

$$\tfrac{1}{9}\frac{A_\alpha}{\omega_{kl,\,\alpha}} + \tfrac{5}{9}\frac{A_\beta}{\omega_{kl,\,\beta}} = 2. \qquad 7.21$$

Suppose that an astrophysical determination of the equivalent breadths of the lines gives W_α and W_β. Then since $W_\alpha^2/W_\beta^2 = \lambda_\alpha^4 f_\alpha \omega_\alpha k_\beta / \lambda_\beta^4 f_\beta \omega_\beta k_\alpha$,

we find $W_\alpha^2/W_\beta^2 = A_\alpha^2\nu_\beta^6 k_\beta/5A_\beta^2\nu_\alpha^6 k_\alpha$ (7.22), taking into account the fact that ω_{kl} is proportional to ν^2.

A_α and A_β can be found from (7.21) and (7.22). The dependence of k_ν on ν has been given by Pannekoek. The method described above is that actually employed by Minnaert. If the P state is not the ground state but a metastable state, the method can be adapted to the case where one of the lines—say $P-D$—is missing, by measuring a line from the ground state instead of the missing line. Referring to the case of *Ca*, illustrated in

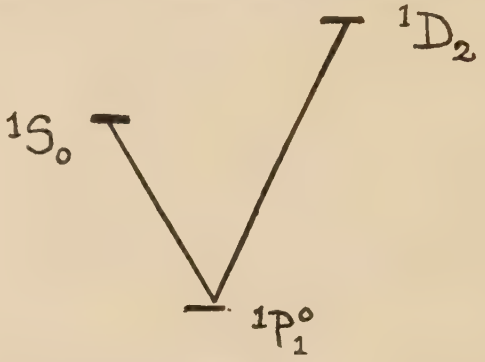

FIG. 3.

fig. 2, let $\mathfrak{A}$ be the sum of the transition probabilities from the state 5^3S_1 to the 4^3P^0 states; let $\mathfrak{B}$ be the sum of the transition probabilities from the level 3^3D_1 to the $^3P^0$ states. Then $\mathfrak{B}$ will also be the sum of the transition probabilities $3^3D_2 \to 4^3P^0$, and also the probability of the transition $3^3D_3 \to 4^3P_2{}^0$. Similarly, let $\mathfrak{C}$ be the sum of the downward transitions for each of the 4^3D states. The individual transition probabilities, in terms of $\mathfrak{A}$ and $\mathfrak{C}$, are shown in Table VIII, which also gives observed equivalent breadths in the solar spectrum obtained with the McClean 3-prism Littrow spectrograph at Cambridge.

TABLE VIII

Wave-length	Multiplet Designation	A.	$f_{abs} \times \omega_{kl}$	ω	Observed W Milliangstroms	$W^2/q_1 f\omega$
6162	$5^3S_1 - 4^3P_2{}^0$	$\frac{5}{9}\mathfrak{A}$	$\frac{1}{9}\mathfrak{A}$	$\mathfrak{A}$	200 ± 13	$72000 \times 1/\mathfrak{A}^2$
6122	$5^3S_1 - 4^3P_1{}^0$	$\frac{1}{3}\mathfrak{A}$	$\frac{1}{9}\mathfrak{A}$		158 ± 9	$75000 \times 1/\mathfrak{A}^2$
6120	$5^3S_1 - 4^3P_0{}^0$	$\frac{1}{9}\mathfrak{A}$	$\frac{1}{9}\mathfrak{A}$		blend	
4457	$4^3D_1 - 4^3P_2{}^0$	$\frac{1}{36}\mathfrak{C}$	$\frac{1}{180}\mathfrak{C}$	$\mathfrak{C}$	57 ± 04	$117000 \times 1/\mathfrak{C}^2$
4456	$4^3D_2 - 4^3P_2{}^0$	$\frac{1}{4}\mathfrak{C}$	$\frac{1}{12}\mathfrak{C}$		blend	
4455	$4^3D_3 - 4^3P_2{}^0$	$\mathfrak{C}$	$\frac{7}{15}\mathfrak{C}$		266(± 04)	$30400 \times 1/\mathfrak{C}^2$
4436	$4^3D_1 - 4^3P_1{}^0$	$\frac{5}{12}\mathfrak{C}$	$\frac{5}{36}\mathfrak{C}$		120 ± 04	$34600 \times 1/\mathfrak{C}^2$
4435	$4^3D_2 - 4^3P_1{}^0$	$\frac{3}{4}\mathfrak{C}$	$\frac{5}{12}\mathfrak{C}$		197 ± 10	$31100 \times 1/\mathfrak{C}^2$
4425	$4^3D_1 - 4^3P_0{}^0$	$\frac{5}{9}\mathfrak{C}$	$\frac{5}{9}\mathfrak{C}$		129	$30000 \times 1/\mathfrak{C}^2$

The quantity $W^2/q_1 f\omega$ is tabulated as it should be constant for any multiplet,* but for the influence of the Doppler effect in broadening the weaker lines, of which 4457 is a good example. We bring in the observed equivalent breadth of the line 4227 by comparing the number of atoms in the 4^1S_0 state with the numbers in the 4^3P^0 states. These numbers stand in a known ratio when the temperature is known. Choosing the optical depth $\tau = 0{\cdot}3$, I find that ϕ has the value 1·0 for the comparison of these states at that depth, that is to say, that the comparison of the equivalent breadths of 4227 and 6162 will give the ratio of the number of atoms in the 4^1S_0 state per c.c. at $\tau = 0{\cdot}3$ to the number of atoms in the $4^3P_2{}^0$ state per c.c. at $\tau = 0{\cdot}3$. This ratio is known from the temperature at $\tau = 0{\cdot}3$, which is 5800° according to § 5. We work backwards and find one further relation between the absorption coefficients, which fills up the gap left by the infra-red multiplet $4^3P^0 - 3^3D$.

We have the relation $W^2 \propto n\lambda^4 f\omega/k_\nu$. Using the suffix α for the line 4227 and β for the line 6162, we get

$$\frac{W_\beta{}^2}{W_\alpha{}^2} = \frac{n_\beta}{n_\alpha}\frac{\lambda_\beta{}^4}{\lambda_\alpha{}^4} \cdot \frac{\frac{1}{9}(\mathfrak{A}/\omega_{kl,\beta})^2}{2\times 2} \cdot \frac{\omega_{kl,\beta}}{\omega_{kl,\alpha}} \cdot \frac{k_\alpha}{k_\beta} = \frac{n_\beta}{n_\alpha} \cdot \frac{\lambda_\beta{}^2}{\lambda_\alpha{}^2} \cdot \frac{1}{36} \cdot \left(\frac{\mathfrak{A}}{\omega_{kl,\beta}}\right)^2 \cdot \frac{k_\alpha}{k_\beta},$$

which gives $\mathfrak{A}$ at once. Similarly we can find $\mathfrak{C}$. The missing $\mathfrak{B}$ can be found from the sum rule applied to any of the 4^3P^0 states. Values of k are taken from Pannekoek's formula, quoted above, which gives relative values of k; at 4227 A., 100; at 4554 A., 77; and at 6162 A., 33.

The results of the computations are

$$\mathfrak{A} = 1{\cdot}15 \times \omega_{kl},$$
$$\mathfrak{B} = 2{\cdot}47 \times \omega_{kl},$$
$$\mathfrak{C} = 1{\cdot}54 \times \omega_{kl}.$$

If these constants were known *a priori* we could use the astrophysical observations to give an independent measurement of the solar temperature. The astrophysical determination of the transition probabilities, as given above, is an approximation, since we have neglected the influence of other transitions, including continuous radiation, from the states concerned. These affect the application of the sum rule.

§ 8. *Calcium Ionisation Temperatures of Late Type Stars*

Contours of 4227 and of *K* for a number of late type stars have been published by Hogg.† These measurements were made with an objective prism spectrograph and, presumably, although the contours cannot lay

* This statement must be taken without prejudice to the writer's argument that the quantity very seldom is constant in a multiplet in the solar spectrum. But it is constant for principal lines, and lines discussed here arise in absorption from a metastable state and behave like principal lines.

† *H.C.O. Bulletin*, 859, 1928.

claim to the highest accuracy, the equivalent breadths will be usable. In those cases where 4227 and K have about the same equivalent breadth, the ratio of the equivalent breadths given even by an objective prism spectrograph will be fairly reliable. In the F and G type stars, where 4227 becomes less than an angstrom in equivalent breadth, the determination with a slitless spectrograph is questionable.

Excluding variable stars, Hogg gives equivalent breadths of the calcium lines in 11 stars whose parallaxes are known. The stellar surface gravity can be estimated by using the mass luminosity relation, which gives the stellar mass, and by determining the stellar radius from the relation

$$R^2 T_e^4 \propto L,$$

assuming a trial value of T_e. With this value of g we can calculate P at $\tau = 0{\cdot}3$, and the observations of equivalent breadth give n_1/n_0 at $\tau = 0{\cdot}3$ exactly as in the case of the Sun. Then n_1/n_0 and P give the first approximation to T_e. If this differs by more than twenty degrees from the trial value, we must make a second approximation, using the first approximation to redetermine R.

There are one or two points to notice. The value of ϕ must be recalculated, ideally, for every fresh temperature; but, as a matter of fact, ϕ maintains the value 0·59 within very close limits even for temperatures as low as 3000°, taking g one-hundredth of the solar value. The constancy of ϕ saves a lot of troublesome computation. Similarly α and k_{3933}/k_{4227} must be calculated for each temperature. As in the case of the Sun, radiation pressure may be neglected, although g is weak. This is because P is so small that k is small. The atmosphere is so transparent that the radiation does little to support it.

There is one theoretical difference between the stellar and solar determinations, namely, that the stellar light is integrated light from the whole disc, while the solar light is taken from the centre of the disc. If there is any differential change of contour between neutral and ionised lines from centre to limb, it must be allowed for in the stellar case. In spite of H. H. Plaskett's observation of a marked change of contour in the *Mg b* group from centre to limb, I have been unable to detect any striking change in 4227 at Cambridge. In these reductions I have supposed that there is no differential change between ionised and neutral lines of a sufficient order of magnitude to affect the determinations.

Table IX shows the results. The visual magnitude, parallax and spectral type are taken from Schlesinger's catalogue.

The stellar temperatures found in this way are considerably higher than some previous estimates. For example, Miss Payne, quoted in Eddington's *Internal Constitution of the Stars*, p. 141, gives $T_e = 5600°$ for type Go, 4000° for type Ko and 3000° for Ma. Russell, Dugan and Stewart adopt about the same values. There is, however, a certain amount of evidence which goes to show that these temperature scales are too low. The temperature 5740° for the Sun, which one sees quoted extensively, is far below the mean of the observed values, and the value 6300° found by the present method

agrees very well with the mean colour temperature of Go stars found by Greaves, Davidson and Martin.

H. H. Plaskett * gives 5000–5500° for a Ko star, β Gem. His determination agrees with those of Coblenz and Wilsing.

TABLE IX

Reduction of Hogg's Measurements

Star	Type	Vis. Mag.	Parallax	Mass ⊙ = 1	Radius ⊙ = 1	Equivalent Breadth in Angstroms		T_e
						4227	K	
			"					°
α Ser	Ko	2·75	0·045	3·0	12·3	5·15	17·6	4730
v Vir	Ko	5·24	·014	3·1	16·6	8·36	15·8	4340
α Hya	K2	2·16	·016	9·8	82·3	6·91	13·4	4090
40 Lyn	K5	3·30	·023	5·6	51·2	9·25	12·3	3970
v Boo	K5	4·28	·012	6·7	56·2	8·30	15·8	4130
λ Dra	Ma	4·06	·018	5·1	43·6	10·86	16·0	4030
δ Oph	Ma	3·03	·029	5·1	46·6	11·10	14·6	3970
13 Sge	Mb	5·56	·008	5·6	46·6	10·16	18·1	4120
β Peg	gMb	2·61	·020	8·5	71·8	8·30	16·4	4120
μ Gem	Ma	3·19	·016	8·1	80·4	10·26	10·8	3800
Boss 3827	Mb	4·86	·007	8·6	80·4	12·30	16·2	3900

§ 9. *Concluding Remarks and Summary*

We have explicitly (§ 3) neglected radiation pressure in the solar atmosphere. The support given by the radiation pressure is feeble compared with solar gravity. Calcium is subject to selective radiation pressure in addition to the general radiation pressure. The point in our argument which is concerned with this circumstance is our assumption that the atmosphere is well mixed, whereas the selective radiation pressure tends to keep calcium higher than other elements of the same atomic weight. A very high degree of stratification would interfere with our calculations, but the turbulent nature of the solar disc argues against such stratification. The chromosphere contains a large amount of ionised calcium, which cannot be counted as well mixed. But the chromospheric Ca^+ is responsible for the narrow line K_3, and not for the broad absorption line K_1. Measurements of an unreversed Ca^+ line are substantially measurements of K_1 and of the Ca^+ in the reversing layer as distinct from the low-pressure chromosphere.

We have followed Russell in assigning a very high percentage of hydrogen in the solar atmosphere. This high percentage of hydrogen has an effect on the coefficient of continuous absorption. But even halving the quantity of hydrogen only increases α by about 10 per cent. If there is less hydrogen than we have assumed, the calcium temperature of the Sun is slightly higher on this account.

* *Pub. D.A.O.*, **2**, 258, 1923.

Summary

A method is developed for determining a precise value of the solar temperature from observations of the equivalent breadth of *Ca* and Ca^+ lines. Using Chandrasekhar's absorption coefficient, the electron pressure P can be calculated at any prescribed optical depth. It is shown that n_1/n_0, the ratio of ionised to neutral calcium atoms per c.c. at this depth, can be found from the equivalent breadths of the principal lines of *Ca* at 4227, and of Ca^+ at 3933 (K), with the help of a certain factor ϕ. It is shown how to calculate ϕ.

Knowing n_1/n_0 and P we can calculate the temperature at the level from Saha's formula. An appropriate modification of Pannekoek's adaptation of Saha's formula is used. The observations of equivalent breadth were made at Cambridge with the McLean Littrow solar spectrograph. The result is $T_e = 6310 \pm 50$ for the integrated solar disc.

In § 7 the absorption coefficients of some neutral calcium lines are calculated from observations made with the same spectrograph. In § 8 the method is applied to Hogg's measurements of 4227 and 3933 in K and M type stars.

ON THE DISSOCIATION OF MOLECULES IN THE ATMOSPHERES OF THE STARS OF THE MAIN SEQUENCE.

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(Communicated by P. A. M. Dirac)

The typical intensity variations of molecular bands in stellar spectra with spectral type and absolute magnitude depend on the corresponding variations with temperature and surface gravity of the total number of molecules in the reversing layer of the star's atmosphere. The determination of this number of molecules, on the assumption of statistical equilibrium, involves problems of the same kind as the calculation of the numbers of atoms in different stages of ionization, the process of ionization being replaced by that of dissociation. In previous treatments of the subject * only the influence of the variation of the dissociation coefficient with temperature was discussed, while the pressure variations were but summarily taken into account. In the following paper an attempt is made to transfer to the dissociative equilibrium problem the method of calculating pressure from general absorption so successfully used by Milne and Chandrasekhar † in

* See especially R. Wildt, *Zs. f. Physik*, 54, 856, 1929, and P. Swings, *Les bandes moléculaires dans les spectres stellaires*, Paris, 1932.

† *M.N.*, 92, 150, 1932. We are very much indebted to S. Chandrasekhar for calling our attention to the possibility of applying this method to the molecular problem.

the ionization equilibrium problem. It will be seen that we obtain in this way a general understanding of the main features of molecular band intensities along the main sequence of spectral types, by assuming for all main sequence stars a similar chemical composition of the atmosphere. It seems, on the other hand, that the spectra of the R–N branch stars, exhibiting strong bands due to carbon compounds, can be interpreted only under the assumption of a different chemical constitution of the stellar atmosphere; this question will be the object of a special investigation.

In the following Section I, we present the theoretical considerations used in the discussion of the main sequence stellar spectra, contained in Section II.

Section I. Theory

1. *Number of Molecules in the Stellar Atmosphere.*—Consider a column of unit cross-section of the reversing layer of the star's atmosphere. Assuming a state of dissociative equilibrium in any infinitesimal layer, of thickness dh, of this column, situated at a depth h from the boundary of the star, we can calculate the fraction ξ of the molecules of the kind considered present in this infinitesimal layer as a function of the temperature and pressure of the layer and of the chemical constitution of the star. If $n(h)$ is the total number of atoms and molecules in a unit volume at depth h, the total number of molecules of the kind considered present in the whole column, of height H, of the absorbing atmosphere will then be given by

$$N = \int_0^H \xi n dh. \tag{1}$$

Now by gravitational equilibrium, and neglecting radiation pressure, we have

$$\frac{dp}{dh} = gn\bar{m}, \tag{2}$$

where p is the total pressure at depth h, and $\bar{m}$ the mean atomic mass of the star's material. If μ_i is the abundance by number of atoms of the element of mass m_i in the limiting case of infinite temperature, then $\bar{m} = \sum_i \mu_i m_i$. We now make the assumption, which we will discuss later, that in each infinitesimal layer the electron pressure $P(h)$ is a constant fraction α of the total pressure $p(h)$. Equation (2) then becomes

$$ndh = \frac{1}{\alpha g \bar{m}} dP, \tag{3}$$

and (1) can be written

$$N = \frac{1}{\alpha g \bar{m}} \int_0^{P_0} \xi dP, \tag{4}$$

where P_0 is the electron pressure at the base of the reversing layer.

We may assume that the temperature of the reversing layer is constant and equal to the effective temperature T of the star. This is the temperature at the optical depth $\tau = \frac{2}{3}$, and we may define the base of the atmosphere as

being situated at an actual depth H corresponding to this optical depth $\tau = \frac{2}{3}$. From this condition we can readily estimate the electron pressure P_0 at the base of the atmosphere. Following Milne and Chandrasekhar we write

$$\tau = \int_0^H \kappa \bar{m} n dh, \tag{5}$$

where the mean absorption coefficient κ is approximately given by

$$\kappa = AT^{-11/2} \frac{\chi^2}{a} Px ; \tag{6}$$

in this formula A is a constant, χ is the mean ionization potential of the atoms contributing to the general absorption, a the mean atomic weight, and x the ionized fraction of these atoms. Assuming this fraction x to be constant, we get by introducing (3) and (6) into (5),

$$\tau = \frac{A}{gT^{11/2}} \frac{\chi^2 x}{a\alpha} \int_0^{P_0} P dP,$$

whence

$$P_0 = \sqrt{\frac{g\tau T^{11/2}}{A}} \cdot \frac{1}{\eta} \tag{7}$$

by putting

$$\eta^2 = \frac{\chi^2}{a} \cdot \frac{x}{2\alpha}. \tag{8}$$

The quantity η is found empirically to be of the order of magnitude of 10 volts ; the constant A, when η is thus expressed in volts, has the value $5{\cdot}62 \cdot 10^{19}$. Taking finally $\tau = \frac{2}{3}$, we can thus calculate P_0 by formula (7) as a function of T and g.

If we determine ξ as a function of the temperature T and of the electron pressure, we obtain from (4) by integration the number of molecules N as a function of T and P_0, and consequently as a function of T and g. The determination of ξ in all cases requires some knowledge of the relative abundances in the star's atmosphere of the elements involved in the reactions considered. No precise information about the chemical constitution of the stars has been as yet derived from observational data,* but it will be seen that the characteristic features of the variation of N with T and g are quite unsensitive with respect to the chemical constitution. It will therefore be permissible to make some simplifying assumptions in general agreement with the evidence obtainable from atomic stellar spectra. We may divide the elements of the periodic system into two groups, the first of which consists of the elements with ionization potentials greater than 11 volts (*i.e.* *H*, *C*, *N*, *O*, the halogens and the rare gases), while the second comprises all the other elements, the ionization potentials of which are about 7 volts. Hydrogen being by far the most abundant element, it would be a fair assumption to take the total number of atoms of the first group as approximately equal to that of the second group. As the elements of the second group

* *Cf.* C. Payne, *Stellar Atmospheres*, chap. xiii, 1925, and also D. Menzel, *Pub. Lick Obs.*, **17**, 217, 1931.

contribute most of the free electrons by ionization in the relevant temperature range (2000°–7000°), we may further assume that the electron pressure is of the same order of magnitude as the partial pressure of all atoms of the second group, which in turn is about the same as the partial pressure of the first group. In other words, we take the fraction α of the order $\frac{1}{3}$. As to the justification of the assumed constancy of α in the temperature range considered, we remark * that for the elements of the first group the ionization varies only by a factor 10 in the whole range, while for the second group the maximum of the ionization falls within that range, so that in this case also its variation is relatively small. Finally, it must be emphasized that the unsensitiveness of our formulæ, which allows us to work under such simplified conditions, imposes at the same time a serious limitation to the significance of any comparison of the theoretical results with observation. In fact, we cannot hope to derive from such a comparison precise values of physical quantities involved in the theory, but we can only expect it to exhibit in a broad way the general consistency of the various observational data on the assumption of a stellar atmosphere in statistical equilibrium.

2. *Dissociative Equilibrium.*—We shall only discuss a special case of dissociative equilibrium which finds application in stars of the main sequence. The well-known equation of dissociative equilibrium of the diatomic molecule AB is

$$\frac{p_A p_B}{p_{AB}} = K_{AB}, \tag{9}$$

where p_A, p_B and p_{AB} are the atomic and molecular partial pressures, and the dissociation coefficient K_{AB} is a function of temperature and of the molecular and atomic constants.

This function is defined by the formula

$$\log K_{AB} = -\frac{5041 D}{T} + \tfrac{3}{2}\log T + \log\left(1 - e^{-\frac{1{\cdot}432\omega}{T}}\right) + \sigma, \tag{10}$$

where D represents the dissociation energy in volts, ω the fundamental vibration frequency in cm.$^{-1}$, and σ is given by

$$\sigma = i + \tfrac{3}{2}\log M + \log G - \log I, \tag{11}$$

or

$$\sigma = i' + \tfrac{1}{2}\log M + \log G - 2\log r. \tag{11'}$$

In these formulæ M is the effective molecular weight (not mass in absolute units), *i.e.* the expression $\dfrac{M_A M_B}{M_{AB}}$, where M_A, M_B, M_{AB} are the atomic and molecular weights ; $G = \dfrac{G_A G_B}{G_{AB}}$, where G_A, G_B, G_{AB} are the statistical weights of the states of the atoms and the molecule taking part in the reaction ; I is the moment of inertia, and r the equilibrium distance of the atoms in the molecule. If I is expressed in g-cm.^{2} multiplied by 10^{40}, the chemical constant i is equal to 6·017 or to 6·318, according to whether the atoms A and B

* *Cf.* E. A. Milne and S. Chandrasekhar, *loc. cit.*

are different or identical; if r is expressed in angstrom units, the constant i' is accordingly equal to 5·8 or 6·1.

Consider now the case of an element A combining with other elements $B_1, B_2, \ldots$ which are so much in excess with respect to A that the decrease in their concentration resulting from the different combinations in which they take part may be neglected. We assume that all atoms and molecules considered are in their normal state. The partial pressures p_{B_i} may thus be written $\mu_{B_i}P$, where μ_{B_i} on our assumptions is the abundance by number of atoms of the element B_i with respect to the group to which this element belongs. If x_i is the fraction of atoms A bound to atoms B_i at depth h, we have

$$p_{AB_i} = x_i \mu_A P, \qquad p_A = \left(1 - \sum_k x_k\right)\mu_A P, \tag{12}$$

and the fraction of molecules AB_i present at depth h in the atmosphere is $\xi_i = x_i \mu_A$. Introducing the values (12) in the equilibrium equations of type (9) we get

$$x_i = \frac{\mu_{B_i} P}{K_{AB_i}}\left(1 - \sum_k x_k\right),$$

whence

$$\xi_i = \frac{\mu_A \mu_{B_i}}{K_{AB_i}} \frac{P}{1 + F_A P}, \tag{13}$$

with

$$F_A = \sum_i \frac{\mu_{B_i}}{K_{AB_i}}. \tag{14}$$

From (4) and (13) we get finally for the total number of absorbing molecules AB_i,

$$N_{AB_i} = \frac{\mu_{B_i}}{K_{AB_i}} \cdot \frac{\mu_A}{a\bar{m}} \cdot \frac{1}{g F_A^2}\{F_A P_0 - ln(1 + F_A P_0)\}. \tag{15}$$

This formula, together with (7), (10) and (14), gives the expression of N_{AB_i} as a function of T and g. When $F_A P_0$ is very small with respect to 1, which is the case for sufficiently high temperatures, this expression reduces to

$$N_{AB_i} \sim \frac{\mu_{B_i}}{K_{AB_i}} \frac{\mu_A}{a\bar{m}} \frac{P_0^2}{2g}, \tag{16}$$

which on account of (7) is independent of g. Appreciable absolute magnitude effects in intensities of molecular bands in this case are thus to be expected only in cold stars. It is easily seen from (15) that these absolute magnitude effects are always *positive* $\left(\frac{dN}{dg} < 0\right)$.

The partial pressure of atoms A is given, according to (12) and (13), by

$$p_A = \frac{\mu_A P}{1 + F_A P}. \tag{17}$$

Now if the ratio $\mu_A / K_{A_2} F_A$ is sufficiently small, the formation of the molecule A_2 will not appreciably influence the equilibria just discussed, and we may calculate in first approximation the fraction ζ of such molecules simply by

introducing in the equation $p_A^2/p_{A_2} = K_{A_2}$ the value (17) of p_A and putting $p_{A_2} = \zeta P$. This gives

$$N_{A_2} = \frac{\mu_A}{K_{A_2}} \frac{\mu_A}{\alpha \bar{m}} \frac{1}{g F_A^2} \left\{ ln(1 + F_A P_0) - \frac{F_A P_0}{1 + F_A P_0} \right\}, \tag{18}$$

and for high temperatures,

$$N_{A_2} \sim \frac{\mu_A}{K_{A_2}} \cdot \frac{\mu_A}{\alpha \bar{m}} \frac{P_0^2}{2g}, \tag{19}$$

again independent of g.

A characteristic feature of formulæ (15) and (18) is the existence, for a given value of g, of a maximum number of molecules for a certain temperature T_m. This temperature does not depend very much on g, but is more sensitive to the value of the dissociation energy. This fact is not without significance for the comparison of this theory with observational data, to which we will presently proceed.

Section II. Applications

1. *General Survey of the Problem.*—It is of course entirely out of question to attack the problem of all possible chemical reactions in a star's atmosphere, even if we avail ourselves of the simplifications due to the fact that the fraction of atoms entering molecular combinations is usually small, and that only diatomic molecules are to be considered, except possibly in very cold stars. The abundance of a given molecule is primarily determined by the abundance of its constituent atoms ; we may therefore expect that the only important compounds will be hydrides, oxides, and nitrogen and carbon compounds. In fact, all compounds which have been so far observed in stars are of these types. In order that the presence of some kind of molecules may be revealed by spectral analysis, it is necessary that the spectrum of this molecule exhibits sufficiently intense bands in the visible region ; among the molecules of the types considered having this property the most important are the oxides *TiO* and *ZrO*, and the carbon compounds *CH*, *CN* and C_2. The visible bands of these compounds are due to absorption from the normal state ; indeed no example is known of the occurrence of visible bands due to absorption from excited states. This is simply due to the fact that the corresponding excitation energies are too high to permit the presence of a sufficient fraction of excited molecules in the temperature range where the number of normal molecules is appreciable ; in the course of the following work we will show this quantitatively in the case of the angstrom bands of *CO*. We shall thus restrict ourselves throughout to the consideration of normal molecules. In all cases which we shall have to treat, these molecules will result from the combination of atoms in the normal state.

By considering only combinations into which abundant elements enter, we further introduce considerable simplifications. In fact, we find ourselves in the conditions of the problem discussed in § 2 of Section I above ; we may neglect all mutual combinations of the abundant elements, assuming constant partial pressures for these elements, and take into account only their combinations with the less abundant elements (and eventually mutual

combinations of the latter). Of course the distinction between abundant and less abundant elements in the sense just explained is to some extent arbitrary. For main sequence stars we shall treat *H*, *O* and *N* as abundant elements and consider their combinations with the less abundant elements *C*, *Ti*, *Zr*. In this case there is some doubt about the relative abundances of *N* and *C*, which seem to be both less abundant than oxygen, and in fact of comparable abundances. This minor uncertainty, however, will not affect our results, the only essential assumption being that oxygen is in excess over carbon. It is thought that this assumption differentiates main sequence stars from the so-called carbon stars of Classes R and N, which would be characterized from our point of view by an excess of carbon over oxygen ; we shall come back to this point later.

The choice of the absolute abundances of the elements considered involves a still greater arbitrariness; but again, as has already been pointed out, this is of no consequence for our purpose. It is only necessary to fix the abundances of the elements *H*, *O* and *N*; we take for simplicity $\mu_H = 1$; further, $\mu_O = 0{\cdot}03$ and $\mu_N = 0{\cdot}001$. As to the combinations we have to take into account, we may of course discard all mutual combinations of the less abundant elements *C*, *Ti* and *Zr*, except C_2, and also the combinations of *Ti* and *Zr* with *H* and *N*. We may thus treat separately : (1) the formation of *TiO*, (2) the formation of *ZrO*, (3) the equilibrium of the carbon compounds *CH*, *CO*, *CN* and C_2.

2. *The Oxides of Titanium and Zirconium.*—The analogous compounds *TiO* and *ZrO*, which are characteristic of Class M and Class S stars respectively, afford the simplest examples of dissociative equilibria in stellar atmospheres. In these cases we have only to apply formula (15), or eventually the approximate formula (16), by putting

$$F_{Ti} = \frac{\mu_O}{K_{TiO}}, \qquad F_{Zr} = \frac{\mu_O}{K_{ZrO}}.$$

An evaluation of K_{TiO} and K_{ZrO} can be derived from Christy's and Lowater's spectroscopic data.* Both molecules are formed in a $^3\Pi$ state probably from an oxygen atom in its normal ^3P state, and a *Ti* or *Zr* atom in its normal ^3F state ; this gives $G = \frac{21 \times 9}{2} = 94{\cdot}5$. The other data are given in the following table :—

	D	ω	$I \cdot 10^{40}$	*M*	
TiO	6·74	1000	51·89	12	Christy's data
	6·96	1008·12	...	...	Lowater's data
ZrO	7·67	936·7	...	13·6	Lowater's data

* For *TiO*, see A. Christy, *Ap. J.*, **70**, 1, 1929, and Lowater, *Proc. Phys. Soc. London*, **41**, 557, 1929. For *ZrO*, see Lowater, *ibid.*, **44**, 51, 1932. See also W. Jevons, *Report on Band Spectra of Diatomic Molecules*, 1932.

The energies of dissociation have not been determined directly, but can be fairly estimated by the Birge-Sponer method of extrapolation. The moment of inertia of *ZrO* is not known; from the empirical formula of Morse, $\omega r^3 \sim$ const., it may be concluded that it is not very different from that of *TiO*. We may thus take the same constant $\sigma = 7{\cdot}8957$ for both reactions, and the only important difference between them is due to the greater energy of dissociation of *ZrO*.

Numerical calculations, the results of which are collected in the following table and represented graphically in fig. 1, have been made for two extreme values of g, viz. $\log g = 0{\cdot}69$ for the giant branch, and $\log g = 5{\cdot}28$ for the dwarf branch:—

T	$\log P_0$		$\log K_{TiO}$	$\log K_{ZrO}$	$\log N_{TiO}$ ‡		$\log N_{ZrO}$ §	
	Giants *	Dwarfs †			Giants	Dwarfs	Giants	Dwarfs
1500	$\bar{2}{\cdot}116$	0·414	$\bar{11}{\cdot}9656$	$\bar{14}{\cdot}8403$	$\bar{5}{\cdot}90$	$\bar{7}{\cdot}61$	$\bar{5}{\cdot}90$	$\bar{7}{\cdot}61$
1800	$\bar{2}{\cdot}334$	0·632	$\bar{7}{\cdot}7996$	..	$\bar{4}{\cdot}12$	$\bar{7}{\cdot}82$		
2000	$\bar{2}{\cdot}459$	0·756	$\bar{5}{\cdot}7250$	$\bar{7}{\cdot}3809$	$\bar{4}{\cdot}17$	$\bar{7}{\cdot}95$	$\bar{4}{\cdot}25$	$\bar{7}{\cdot}95$
2500	$\bar{2}{\cdot}726$	1·023	$\bar{1}{\cdot}1987$	$\bar{3}{\cdot}3234$	$\bar{6}{\cdot}30$	$\bar{7}{\cdot}67$	$\bar{5}{\cdot}92$	$\bar{6}{\cdot}22$
3000	$\bar{2}{\cdot}944$	1·241	1·5220	$\bar{1}{\cdot}9593$	$\bar{8}{\cdot}31$		$\bar{7}{\cdot}87$	$\bar{7}{\cdot}76$
3500	$\bar{1}{\cdot}128$	1·425	3·1884	1·8426	$\bar{9}{\cdot}01$	Independent of g	$\bar{8}{\cdot}36$	Independent of g
4000	$\bar{1}{\cdot}287$	1·584	4·4397	3·2677	$\bar{10}{\cdot}08$		$\bar{9}{\cdot}25$	
4500	$\bar{1}{\cdot}428$	1·725	5·4171	4·3753	$\bar{11}{\cdot}38$		$\bar{10}{\cdot}42$	
5000	$\bar{1}{\cdot}554$	1·851	6·2114	5·2738	$\bar{12}{\cdot}84$		$\bar{11}{\cdot}78$	
5500	$\bar{1}{\cdot}668$	1·965	6·8458	5·9934	$\bar{12}{\cdot}43$		$\bar{11}{\cdot}29$	
6000	$\bar{1}{\cdot}772$	2·068	7·3835	6·6021	$\bar{12}{\cdot}10$		$\bar{12}{\cdot}89$	
6500	$\bar{1}{\cdot}867$	2·164	7·8402	7·1190	$\bar{13}{\cdot}84$		$\bar{12}{\cdot}56$	
7000	$\bar{1}{\cdot}956$	2·253	8·2330	7·5633	$\bar{13}{\cdot}62$		$\bar{12}{\cdot}29$	

* $\log g = 0{\cdot}69$.
† $\log g = 5{\cdot}28$.
‡ Factor $\mu_{Ti}/\alpha\bar{m}$ omitted.
§ Factor $\mu_{Zr}/\alpha\bar{m}$ omitted.

The general features of these results may be compared with the observational evidence concerning Class M and S stars. First of all we see that the variation of N_{TiO} is in satisfactory agreement with the temperature scale of Class M. A rather steep increase of the number of *TiO* molecules begins in the region about 3500°, which corresponds to the limit of visibility of *TiO* bands in stellar spectra. The number of *TiO* molecules goes on increasing till temperatures about 2000°: this fits in very well with the classification of M-type spectra according to increasing intensities of *TiO* bands with increasing redness of the stars, the latest M-type spectra of long-period variables such as Mira Ceti corresponding to temperatures of the order of 2000°.

Consider now the relative situations of the *ZrO* and *TiO* curves, which are principally determined by the shift of the *ZrO* maxima towards higher temperatures, owing to the greater dissociation energy of *ZrO*; the maxima of giant branch curves are sharper than those of dwarf branch curves. From

these features it is possible to understand that, for a properly chosen value, or range of values, of the ratio of the abundances of *Ti* and *Zr*, the *ZrO* bands would not be likely to be visible in any dwarf spectrum, while they could occur in giant spectra of early M type with an intensity comparable with, or even greater than that of the *TiO* bands. In fact,* stellar spectra of Class S, which are distinguished by the presence of *ZrO* bands, closely resemble early M-type spectra with giant characteristics; most of them are indeed analogous to long-period variable Md type spectra. This suggests the possibility of interpreting Class S (so far as the presence of *ZrO* is concerned)

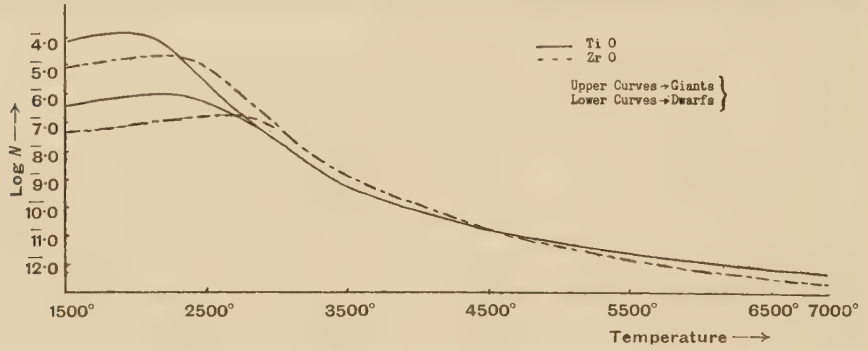


FIG. 1.—*Numbers of* TiO *and* ZrO *Molecules* ($\mu_{Ti}/\mu_{Zr}=10$).

not as a branching off the main sequence, but simply as a part of early gM class. Cases where the *TiO* bands are very much fainter than the *ZrO* bands might be accounted for by a relatively higher abundance of *Zr*.

Absolute magnitude effects might have some influence on the variation of intensity of the *TiO* bands during a period of a Md-type variable star, for if we accept the pulsation theory, the surface gravity of the star will vary periodically as well as the temperature. The kind of effects which might thus be expected will be illustrated by the example of Mira Ceti, for which we possess sufficient numerical data.† Assuming a mass equal to five times that of the Sun, we have for the different phases :

Brightness	T	Radius	log g
Maximum	2400°	354	0·0387
Intermediate	2050	316	0·1373
Minimum	1800	354	0·0387
Intermediate	2150	392	$\bar{1}$·9501

The resulting variation of the number of *TiO* molecules during a period is illustrated in fig. 2 : we find in agreement with observation a continuous

* For the description of M and S classes, see, for example, R. H. Curtiss, *Handbuch der Astrophysik*, V/1, 89 *et seq.*, 1932. See also C. Payne, *Stars of High Luminosity*, p. 230, 1930.

† A. Joy, *Ap. J.*, 63, 281, 1926.

increase of intensity of the *TiO* bands as the star's luminosity decreases, and conversely (if we disregard the slight shift of the band intensity maximum with respect to the luminosity minimum) a decrease of *TiO* band intensity with increasing brightness of the star. It may be noted that temperature variation alone without change of radius would result in a different behaviour of the bands, since along a constant surface gravity curve the band intensity would pass through a maximum during each half-period; but of course no great significance can be attributed to this remark in the

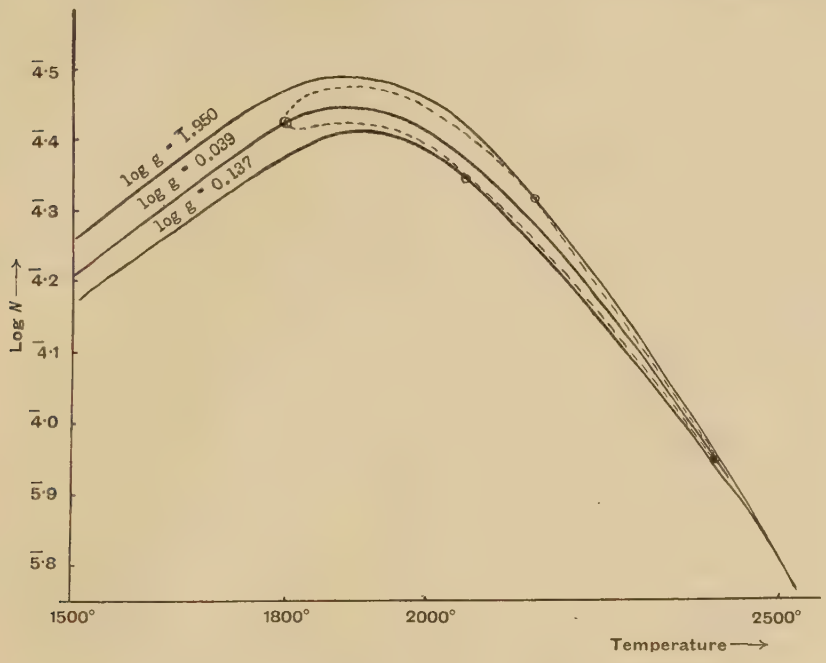


FIG. 2.—*Number of* TiO *Molecules in the Atmosphere of Mira Ceti.*

particular instance considered, where only small quantitative differences are involved.

3. *The Carbon Compounds.*—We now pass to the application of our formulæ to the carbon compounds *CH*, *CO*, *CN* and C_2. The abundant elements B_1, B_2, B_3 will be identified with *H*, *O*, *N* respectively; the less abundant element "*A*" is carbon. We must first evaluate the dissociation coefficients of the four reactions, for which fairly accurate spectroscopic data are available.* We may apply formula (10) with the values of the constants given in the first table on p. 720.

Except in the case of *CN*, the spectroscopic values of *D* agree with the thermochemical determinations;† for *CN* the thermochemical value has been preferred to the value of 7 volts derived in a very indirect way from spectroscopic considerations; it seems that the thermochemical value is not

* See R. Mulliken, *Reviews of Modern Physics*, 4, 1, 1932 [table xviii, and pp. 14 (*CH*, *CO*), 66 (C_2), 67–68 (*CN*)].

† H. G. Grimm and H. Wolff, *Handbuch der Physik*, 24, 536, 1927.

	D Volts	ω cm.$^{-1}$	r A.	M	G	σ
CH	4	2930	1·12	12/13	$\frac{9\times 2}{2}=9$	6·6385
CO	10	2167	1·15	48/7	$\frac{9\times 9}{1}=81$	8·0051
CN	(9·2)	2069	1·17	84/13	$\frac{9\times 4}{2}=18$	7·3240
C_2	5·5	1642	1·31	6	$\frac{9\times 9}{2}=40{\cdot}5$	7·8630

in contradiction with these considerations. The states of the atoms and molecules have been taken as follows :—

$$H(^2S),\ O(^3P),\ N(^4S),\ C(^3P);\ CH(^2\Pi),\ CO(^1\Sigma),\ CN(^2\Sigma),\ C_2(^3\Pi).$$

The constants σ have been calculated by (11′). For the dissociation coefficients we get :

T	log K_{CH}	log K_{CO}	log K_{CN}	log K_{C_2}
2000°	0·9670	$\bar{1}3{\cdot}2178$	$\bar{1}1{\cdot}5370$	$\bar{2}{\cdot}3441$
2500	3·5307	$\bar{8}{\cdot}7960$	$\bar{7}{\cdot}7104$	1·6542
3000	4·9592	$\bar{4}{\cdot}2330$	$\bar{4}{\cdot}8769$	3·5624
3500	5·9863	$\bar{2}{\cdot}6945$	$\bar{1}{\cdot}1445$	4·9474
4000	6·7612	0·5448	0·8498	5·9732
4500	7·3675	1·9871	2·2294	6·7815
5000	7·8552	3·1439	3·2456	7·4316
5500	8·2564	4·0922	4·1200	7·9651
6000	8·5926	4·8842	4·8503	8·4114
6500	8·8785	5·5542	5·4700	8·6905
7000	9·1243	6·1328	5·9780	9·1166

Consider first equation (16), valid for high temperatures. The ratio of the numbers of molecules AB_1 and AB_2 at a given temperature is then simply given by

$$\frac{N_{AB_1}}{N_{AB_2}}=\frac{\mu_{B_1}}{\mu_{B_2}}\frac{K_{AB_2}}{K_{AB_1}}, \tag{20}$$

a formula which may be useful in order to estimate the abundance ratio μ_{B_1}/μ_{B_2} from a known intensity ratio of molecular bands. It has been observed,* for instance, that the CH and CN bands have the same upper limit of visibility, at F8, which corresponds to a temperature of about 6500°. Comparing the K values for this temperature, we see that this simultaneous disappearance implies a ratio μ_H/μ_N of the order 10^3; in this estimation,

* P. Swings and O. Struve, *Phys. Rev.*, **39**, 142, 1932. These authors develop considerations analogous to those of the text, but based on (9) instead of (20).

which is in accordance with our assumed absolute abundances $\mu_H = 1$, $\mu_N = 0{\cdot}001$, allowance must be made for a possibly large value of the ratio of the absorption probabilities of *CH* and *CN*.

Comparing by the same formula the numbers of molecules *CO* and *CN* at the same temperature, we find about 30 times more *CO* than *CN*; but from these *CO* molecules only a fraction of the order of 10^{-6} is in the state of 8-volt excitation energy giving rise to the angstrom bands; it is therefore not to be expected that these bands would appear in any stellar spectrum with sufficient intensity.

As in the case of *TiO* and *ZrO* a calculation of the numbers of molecules *CH*, *CN* and C_2 by formulæ (15) and (18) has been carried out for $\log g = 0{\cdot}69$ and for $\log g = 5{\cdot}28$. The results are collected in the following table and illustrated by fig. 3:—

T	$\log N_{CH}$ *		$\log N_{CN}$ *		$\log N_{C_2}$*	
	Giants †	Dwarfs ‡	Giants	Dwarfs	Giants	Dwarfs
2000°	$\bar{1}5{\cdot}54$	$\bar{1}7{\cdot}24$	$\bar{7}{\cdot}97$	$\bar{9}{\cdot}67$	$\bar{2}\bar{5}{\cdot}79$	$\bar{2}\bar{9}{\cdot}27$
2500	$\bar{1}2{\cdot}82$	$\bar{1}4{\cdot}52$	$\bar{5}{\cdot}64$	$\bar{7}{\cdot}34$	$\bar{1}7{\cdot}25$	$\bar{2}2{\cdot}85$
3000	$\bar{1}0{\cdot}94$	$\bar{1}2{\cdot}72$	$\bar{4}{\cdot}02$	$\bar{7}{\cdot}80$	$\bar{1}3{\cdot}48$	$\bar{1}7{\cdot}45$
3500	$\bar{9}{\cdot}26$	$\bar{1}0{\cdot}28$	$\bar{5}{\cdot}10$	$\bar{6}{\cdot}12$	$\bar{1}2{\cdot}29$	$\bar{1}4{\cdot}47$
4000	$\bar{1}0{\cdot}82$	$\bar{1}0{\cdot}74$	$\bar{7}{\cdot}73$	$\bar{7}{\cdot}65$	$\bar{1}3{\cdot}61$	$\bar{1}3{\cdot}44$
4500	$\bar{1}0{\cdot}50$	Independent of g	$\bar{8}{\cdot}64$	Independent of g	$\bar{1}3{\cdot}08$	Independent of g
5000	$\bar{1}0{\cdot}26$		$\bar{9}{\cdot}87$		$\bar{1}4{\cdot}69$	
5500	$\bar{1}0{\cdot}09$		$\bar{9}{\cdot}22$		$\bar{1}4{\cdot}48$	
6000	$\bar{1}\bar{1}{\cdot}96$		$\bar{1}0{\cdot}70$		$\bar{1}4{\cdot}14$	
6500	$\bar{1}1{\cdot}87$		$\bar{1}0{\cdot}27$		$\bar{1}4{\cdot}05$	
7000	$\bar{1}1{\cdot}79$		$\bar{1}1{\cdot}94$		$\bar{1}5{\cdot}86$	

* Factor $\mu_C/\alpha\bar{m}$ omitted; $\mu_C \sim 10^{-4}$. † $\log g = 0{\cdot}69$. ‡ $\log g = 5{\cdot}28$.

A most striking feature of these results is the total disappearance of C_2 at low temperatures (from about 3000°). This is due to the fact that in presence of an excess of oxygen the formation of the stable oxide *CO* dominates the equilibrium variations of all carbon compounds in this temperature region. The characteristic coefficient F_C is indeed practically equal to $\frac{\mu_O}{K_{CO}}$ for these temperatures. It would only be possible for C_2 to subsist at such low temperatures if carbon were in considerable excess over oxygen; in this case, however, no oxide would be present in appreciable quantity. We thus see, in particular, that the presences of C_2 and *TiO* are mutually exclusive at low temperatures.* This corresponds to a well-known peculiarity of stellar spectra. While M-type spectra of main sequence red stars, characterized by *TiO* bands, do not exhibit C_2 bands, conversely the latter

* This behaviour has been qualitatively described by R. H. Curtiss (see *Handbuch der Astrophysik*, V/1, 97, 1932) in the same connection.

bands form a most conspicuous feature of the R- and N-type spectra of carbon stars, from which *TiO* bands are absent. Of course the possibility is left open, at temperatures about 3000°—where *TiO* concentration begins to increase, and C_2 concentration to decrease sharply—for some star to show bands of both compounds in its spectrum. This may be the case for α Herculis (Mb), in whose spectrum Stebbins found indications of C_2 bands.*

The general behaviour of *CN*, as regards both temperature variation and absolute magnitude effects, is in good agreement with Lindblad's observations, on which Curtiss reports as follows:† "The development of these

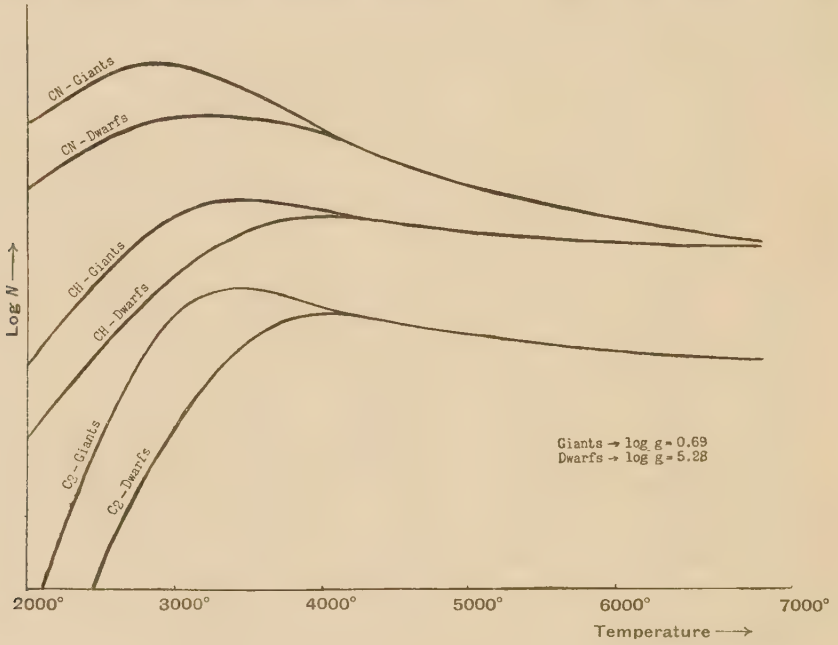


FIG. 3.—*Numbers of Molecules* CH, CN *and* C_2.

bands is at a maximum for giant stars between G5 and K5 and diminishes strongly towards the F and M types. In dwarf stars of all types the bands are only faintly developed."

The evolution of *CH* is parallel to that of *CN*, the maximum being somewhat displaced towards higher temperatures; for the *G* band Rufus‡ indicates, in fact, a maximum at *K*0. Positive absolute magnitude effects in Cepheid variables are noted by Miss Payne.§ In contrast to C_2 bands, both *CN* and *CH* bands appear in early M-type spectra.‖

* *Lick Obs. Bull.*, No. 41, 1903. The identification is not certain; see R. Wildt, *loc. cit.*, p. 865.

† *Handbuch der Astrophysik*, **V/1**, 84, 1932.

‡ *Publ. Obs. Michigan*, **3**, 258, 1923.

§ C. Payne, *Stars of High Luminosity*, p. 214, 1930. P. Swings (*M.N.*, **92**, 140, 1931) found, however, a stronger absorption of *CH* in the Sun than in δ Cephei.

‖ See R. Wildt, *loc. cit.*

In conclusion a few words may be added on the direct photometric determination by Swings * of the relative numbers of *CH* molecules at different phases of δ Cephei. Swings found the following variation of N_{CH} with spectral type :—

Spectral Type	N_{CH}
G6	100
G5	72
G3	39

This change is due to temperature variation only, as N_{CH} does not appreciably depend on g in the temperature range of Class G. A comparison of the above relative values of N_{CH}, with the calculated values for temperatures around 4700° (average temperature of Class gG5),† leads to the attribution of the following temperatures to the spectral types considered at the different phases of δ Cephei :—

G6	4400°
G5	4640
G3	5300

These values, the determination of which is largely independent of our assumptions on the chemical composition of the stars, are in good agreement with the average estimates of temperatures of F, G and K giant and Cepheid stars.

Our thanks are due to P. Swings, and especially to S. Chandrasekhar, for valuable discussions.

Cointe Observatory, Liége University:
1933 *June* 1.

* P. Swings, *M.N.*, **92**, 140, 1931.
† Swings compared them with the K_{CH} values, neglecting pressure variation.

THE DISSOCIATION OF MOLECULES IN THE ATMOSPHERES OF THE CARBON STARS.

L. Rosenfeld.

(Communicated by P. A. M. Dirac)

In a previous paper * it was shown that the general features of the intensity variations of molecular bands in spectra of the main sequence stars could be accounted for on the theory of dissociative equilibrium in a stellar atmosphere of idealised chemical composition and state of ionisation. In particular it appeared that from the point of view of chemical composition the so-called carbon stars of Classes R and N must differ from the main sequence stars by the abundance of oxygen with respect to carbon, this abundance being large in the latter and small in the former. It will be shown in the present note that this difference alone seems sufficient to explain the peculiar behaviour of the bands of carbon compounds in the spectra of R and N types.

1. *Theory.*—We will retain for the atmospheres of the carbon stars all the assumptions regarding chemical constitution and ionisation on which the theory of main sequence stars was based, with the only exception of the value of oxygen abundance. We assume oxygen for some unknown cause to be so scarce in the atmosphere of a carbon star that its influence on the chemical equilibria of the more abundant elements H, N and C may be neglected in first approximation. We have thus to apply in the present case exactly the same formulæ as in our previous work, but for the fact that the term μ_O/K_{CO} must be omitted from the characteristic coefficient F_C.

In this way we obtain for the numbers of molecules CH, CN and C_2 in the atmosphere as functions of temperature and surface gravity curves of the same general type as in the previous case. It may be observed that this type of curve is very general indeed, and would not be essentially modified, for instance, by the consideration of more complicated reactions involving polyatomic compounds. The specification of definite conditions for the reaction partners determines primarily the temperature scale of the curves, and may also affect to some extent the rate of variation of the molecule numbers. Thus, comparing our present problem with that of the main sequence stars, we still expect here the occurrence of maxima for the different molecule numbers, and a sharp decrease of C_2 with decreasing temperature, but since the dominant compound CO is now replaced by CN, which has a lower dissociation energy, these features will be shifted towards lower temperatures. It further happens that the CN curves are much flatter at low temperatures when oxygen is absent. We shall see presently how these differences are related to observed differences in the behaviour of the corre-

* Y. Cambresier and L. Rosenfeld, *M.N.*, 93, 710, 1933.

sponding band intensities. At high temperatures the numbers of molecules CH, CN and C_2 become independent of F_C, and thus of the oxygen abundance ; this explains of course the gradual merging of the N–R sequence into the main sequence with increasing temperature.

Since all known stars of Classes R and N are giants, many of them even supergiants of low mass, numerical calculations of the molecule numbers have been carried out for two values of the surface gravity g, viz. $\log g = 2{\cdot}90$ and $\log g = \bar{3}{\cdot}90$, representative respectively of ordinary giants of high temperature and of supergiants of low mass. The results are contained in the following table and illustrated in the accompanying diagram:—

T	$\log N_{CH}$*		$\log N_{CN}$*		$\log N_{C_2}$*	
	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)
2000°	$\bar{1}0{\cdot}73$	$\bar{1}2{\cdot}23$	$\bar{1}{\cdot}16$	$\bar{4}{\cdot}66$	$\bar{1}5{\cdot}80$	$\bar{1}9{\cdot}01$
2500	$\bar{8}{\cdot}39$	$\bar{1}0{\cdot}11$	$\bar{1}{\cdot}21$	$\bar{4}{\cdot}93$	$\bar{1}0{\cdot}06$	$\bar{1}5{\cdot}66$
3000	$\bar{9}{\cdot}94$	$\bar{9}{\cdot}67$	$\bar{3}{\cdot}02$	$\bar{4}{\cdot}75$	$\bar{1}1{\cdot}34$	$\bar{1}2{\cdot}79$
3500	$\bar{9}{\cdot}28$		$\bar{5}{\cdot}12$		$\bar{1}2{\cdot}32$	
4000	$\bar{1}0{\cdot}82$		$\bar{7}{\cdot}73$		$\bar{1}3{\cdot}61$	
4500	$\bar{1}0{\cdot}50$		$\bar{8}{\cdot}64$		$\bar{1}3{\cdot}08$	

(*a*) $\log g = \bar{3}{\cdot}90$ (*b*) $\log g = 2{\cdot}90$

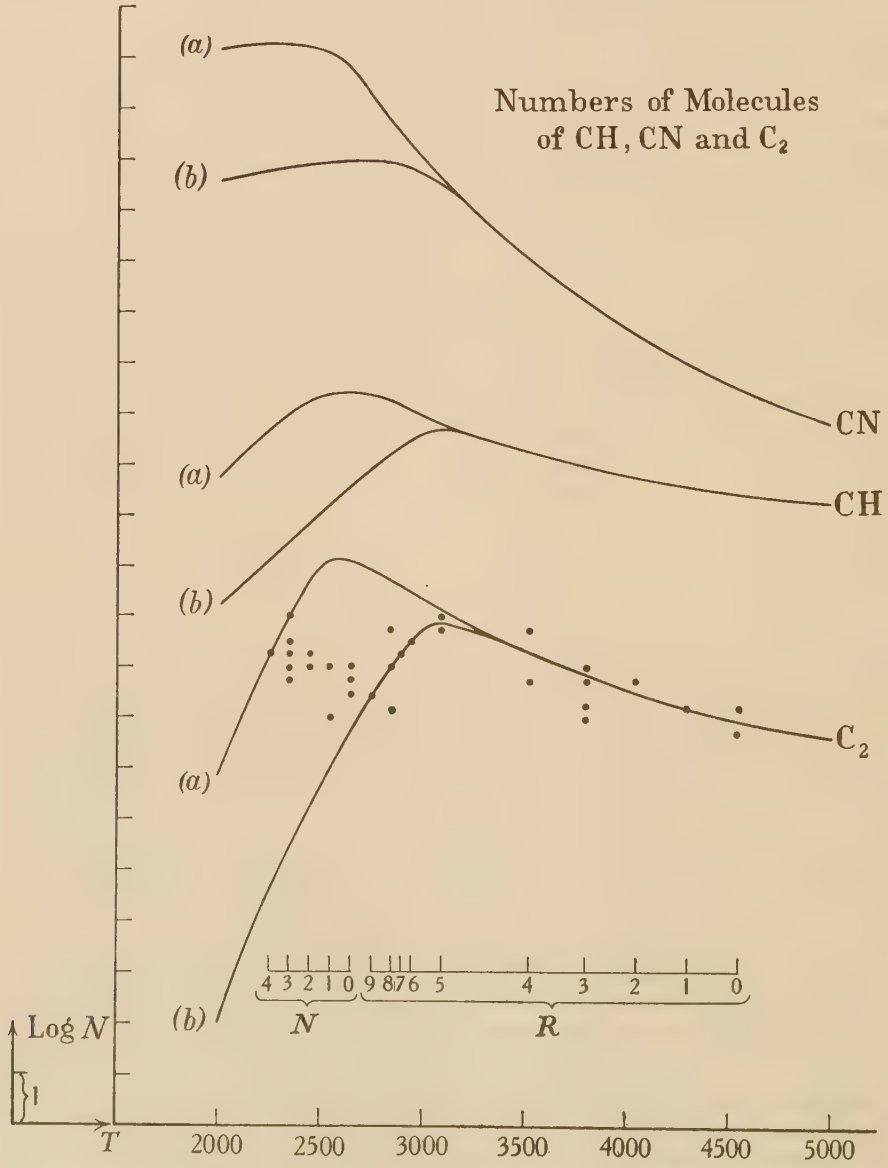
* Factor $\frac{\mu_C}{a\bar{m}}$ omitted ; $\mu_C = 10^{-4}$

2. *Discussion of Observational Data.*—Passing now to the comparison of the theoretical results with the observational data, we follow the generally accepted view, established by Rufus,* that the effective temperature of the stars decreases along the sequence G–R–N. Our task will consist in comparing the observed intensity variations of the bands due to the different carbon compounds with the calculated variations of their numbers of molecules. We shall see that both variations can be made to fit together and yield a plausible temperature scale for the sequence R–N.

In his detailed study of spectra of carbon stars, Shane † stresses the fact that the observed band intensities, if plotted according to spectral sequence, exhibit a strong scattering, which is specially marked for the subdivisions of Class N and the late subdivisions of Class R. This scattering clearly shows the insufficiency of a one-dimensional temperature classification. Now surface gravity obviously offers itself as the second parameter necessary to account for the observed scattering of spectral features. From the calculated curves we see that surface gravity becomes operative in this respect only at low temperatures, up to about 3000°. A temperature range about 3000° seems plausible from general spectral evidence for the late subdivisions of Class R. Now, as we shall see presently in more detail,

* W. C. Rufus, *Publ. Obs. Univ. Michigan*, **2**, 103, 1916.
† C. D. Shane, *Lick Obs. Bull.*, **13**, 123, 1928.

intensity maxima of the bands due to carbon compounds are observed to occur precisely in that spectral range. On the other hand, we notice that the maximum molecule numbers of these compounds occur in the temperature



range about 3000° for the larger values of g which are to be considered. From this we infer that the curves corresponding to these larger g-values must be representative of Class R stars, while the g-values of Class N stars might be more widely scattered toward lower values, in conformity with the supergiant character of the latter stars. In the following discussion we shall therefore use the calculated (b)-curves, *i.e.* those corresponding to

$\log g = 2{\cdot}90$, to determine more precisely the temperature scale of Class R. Naturally no undue stress should be laid on the quantitative aspect of this discussion.

A most prominent spectral feature of carbon stars is the very strong absorption by cyanogen, corresponding to the large abundance of this compound with respect to the others. In fact, cyanogen bands may be traced throughout the sequence R–N, and Rufus * says that they "appear to be somewhat stronger in the spectra of Class N stars, especially λ 4553·6 and λ 4606." Shane,† however, distinguishes between two groups of *CN* bands, which according to him behave in rather different ways, so that the 4606 A. group would not be typical of the entire cyanogen spectrum. But this conclusion, which is not easily understandable from a physical point of view, has since been criticised by Wurm ‡ by purely observational arguments. In consequence, we can disregard Shane's results about his so-called first group (bands at 4216 A., 3883 A. and 3590 A.) and adopt as representative of the observed variation of *CN* absorption his description of the behaviour of the 4606 A. group : "The intensities of these bands do not vary rapidly from one spectral subdivision to another. An examination of 17 plates . . . reveals a noticeable increase in their strength as far as Class R8, beyond which they are nearly constant through Classes No, N3 and N5." The constancy (or even increase, according to the remark of Rufus cited above) of *CN* band intensity through Class N is in striking agreement with the flatness of the calculated molecule number curves. The maximum intensity at R8 leads by comparison with the theoretical (*b*)-curve to a temperature of about 2800° for this spectral subdivision.

A satisfactory agreement of theory with observation is also found in the case of *CH*. Rufus § indicates in a general way for the G band a similar variation through G–R–N sequence as through the G–K–M or main sequence. Shane || goes into more detail, and notes "an increase of the intensity of G from Ro to R3 where it appears with unusual strength. Beyond R3 its intensity decreases slowly at first and then more rapidly until in N3 it is nearly or quite absent." He adds that the existence of some irregularities "combined with the small number of stars examined causes some uncertainty in the position of maximum development, but there is not much doubt that it lies somewhere in the range R2–R5." The temperature of R8 being fixed at 2800°, the position of the maximum of the calculated (*b*)-curve at 3100° is compatible with the observed range of maximum intensity of the G band.

We have now to discuss the Swan bands representative of C_2. These bands were specially examined by Shane,¶ whose result, despite an apparent discrepancy, is compatible with the previous observations of

* W. C. Rufus, *Publ. Obs. Univ. Michigan*, **2**, 103, 1916.
† C. D. Shane, *Lick Obs. Bull.*, **13**, 123, 1928.
‡ K. Wurm, *Zeitsch. f. Astroph.*, **5**, 260, 1932.
§ W. C. Rufus, *Publ. Obs. Univ. Michigan*, **3**, 258, 1923.
|| C. D. Shane, *loc. cit.*
¶ C. D. Shane, *loc. cit.*

Rufus.* Shane states further that all the other observed C_2 bands show a behaviour similar to that of the Swan bands, which may thus be safely taken as characteristic of the whole spectrum. There is an accentuated intensity maximum at R5 followed by a rapid decrease till No, with large scattering at R8. From No the intensity starts again increasing, individual values being strongly scattered at N3. While the maximum at R5 closely corresponds to the calculated maximum of the theoretical (*b*)-curve at 3050°, the later increase of intensity through Class N could not possibly be accounted for by any constant surface gravity curve. This peculiarity already led Shane to the conclusion that "temperature may not be of prime importance in determining the spectrum of the carbon stars." But if we assume for Class N stars a sufficient range of variation of surface gravity toward low values, it becomes possible to distribute the molecule numbers deduced from the observed intensity values between two constant g curves such as our calculated (*b*)- and (*a*)-curves. A glance at these theoretical curves then shows that in spite of the sharp decrease of the number of C_2 molecules for each value of g, they would afford sufficient room for an increase of the molecule number of the observed type through a temperature range extending over 500°. In order that the number of molecules in this range be of the same order of magnitude as for early R-type stars, the mean temperature of Class N should be taken about 2500°.

Our method of attack cannot yield more than such simple qualitative suggestions, and it would of course be futile to try to derive more definite results. Nevertheless a closer discussion of Shane's observations for the purpose of testing the general consistency of the above interpretation might not be out of place. An attempt has therefore been made to plot the molecule numbers obtained from the observed intensities on the theoretical diagram (see figure). Shane's intensity scale has been reduced to molecule numbers by fixing on the (*b*)-curve the subdivision Ro (mean intensity 2) at 4500°, and the subdivision R5 (mean intensity 9·5) at 3100°. The temperature of other spectral subdivisions up to No have then been found from the observed mean intensities by following the (*b*)-curve. For the subdivisions of Class N equal temperature intervals of 100° have been assumed. In this way we get for Class N a temperature range from 2650° to 2250°. These temperatures are decidedly higher than the estimates (1500°–1700°) recently derived by Wurm † from a study of the relative intensities of the different C_2 bands in each individual spectrum. But in view of the many uncertainties, both theoretical and observational, of the problem, this discrepancy does not seem to be serious in this connection.‡

3. *Conclusion.*—Our general conclusion will thus be that the observed intensity variations of molecular bands in stellar spectra may be accounted

* W. C. Rufus, *Publ. Obs. Univ. Michigan*, **2**, 103, 1916.

† K. Wurm, *Zeitsch. f. Astroph.*, **5**, 260, 1932.

‡ It has to be remembered in this connection that a similar discrepancy exists even for the case of the Sun, for which Birge estimates a temperature of 4000° from the observations on the relative intensities in the *CN* bands, while of course the generally accepted temperature for the reversing layers is nearer 5500°.

for in a general way on the theory of dissociative equilibrium by assuming an idealised chemical composition of the stars' atmospheres, uniform for all stars of the main sequence on the one hand, and for all carbon stars on the other hand, the two sequences differing only by the ratio of carbon and oxygen abundances.

Cointe Observatory, Liége University:
1933 *July* 4.

VARIATIONS IN THE SPECTRUM OF 42 CAMELOPARDALIS.

D. L. Edwards, A.R.C.S., D.I.C.

(Communicated by the Director of the Norman Lockyer Observatory)

The group of short-period variables of spectral type B, usually called "β Canis Majoris variables" (from one of the earliest members of the group to be discovered), forms an interesting field of study from the spectroscopic point of view which has not yet been adequately explored. They are usually grouped together with the Cepheid variables on account of their short periods and the shapes of the velocity curves, but certain peculiarities (in addition to the early spectral type) tend to differentiate them from the Cepheids and require further explanation. Three types of variation may be associated with these stars, namely, in radial velocity, in magnitude, and in width or intensity of the spectral lines. The first of these has been the most studied and usually forms the basis for the inclusion of a star in the group. The variations are not always constant, either in period or amplitude, and there may be an additional variation of long period indicating orbital motion (which may account for such changes of amplitude or length of the short period). Variations in magnitude may or may not be present, and when detected have not always proved to possess exactly the same period as the radial velocity variations. Even when the period of the latter is found to change, the magnitude variations appear to be at least approximately constant in period. Variations in line intensity and width have not received very great attention; but here again there are indications that though the period (when variations are present) is nearly equal to the short radial velocity period, it is not necessarily the same and is usually constant.

This apparent independence of the three types of variation—one or more of which may even be absent altogether—suggests that a study of such stars from the point of view of line intensity changes alone might be of value, especially as this aspect of the problem has not hitherto received so much attention. The equipment at the Norman Lockyer Observatory, while not suitable for radial velocity work, is well adapted for the study of changes in line intensities, and a number of β Canis Majoris variables (known or suspected) have been placed on the working list for special investigation.

The first star for which definite results have been obtained is 42 Camelopardalis (H.D. 48879, position in 1900 R.A. $6^h\ 40^m{\cdot}5$ Dec. $+67°\ 41'$, mag.

5·04, spectral type B3). This was not previously known definitely to be a β Canis Majoris variable, but was included by Henroteau * in his list of such stars as a "suspect"—apparently on account of the Yerkes radial velocity measurements, though no confirmation of variability has yet appeared. These Yerkes observations † were the first velocities to be published. The star was announced as a possible spectroscopic binary, and some of the lines were suspected of being double in some of the measures. This doubling, however, has not since been confirmed, and in a later list of radial velocities of Helium stars ‡ the earlier measures are given without any remarks. Only five plates were measured over an interval of more than three years ; so that a short-period change, if present, would probably not have been detected. The next observations were made at Ottawa by Henroteau,§ who took three plates on one night, which gave only a small range in velocity (from +2 to +16 km./sec.). The author states that the star was rather faint to study with the spectrograph used, but the spectrum had good lines for measurement. Also that "On the last two plates of the Yerkes Observatory, Lee suspected the lines to be double. Our spectrograms do not suggest this ; it is, however, very likely that 42 Camelopardalis is a short-period binary, and if it is of the β Canis Majoris type variations of line width are to be expected." Plaskett and Pearce,‖ from observations made at Victoria, give the following remarks: "Hydrogen lines are broad and winged. The Stark line λ 4770 is suspected. Announced as a binary, Y.O. . . . but the range is small and the variability doubtful."

The above observations are somewhat inconclusive, but are not inconsistent with the possibility of a short-period variation. In view of the line intensity changes described later, the star might be worth investigating afresh for radial velocity changes.

Photometric observations are very scarce, but R. H. Baker ¶ states that the light was constant on one night in 1923 at the Lick Observatory. No records of changes in line intensities or widths have been published other than the suspected doubling of lines on the Yerkes spectrograms.

The Sidmouth photographs were taken with the 12-inch McClean refractor of focal length 134 inches and objective prism of 20°, giving a dispersion of about 32 A. per mm. at $H\gamma$. Barnet "Super Press" plates were used, and good spectra were obtainable in from 10 to 20 minutes according to the state of the atmosphere and the altitude of the star. At the beginning of the series photographs were taken at intervals of 30^m (between the times of mid-exposure), but as the extreme shortness of the period became evident this interval was reduced to 20^m in order to obtain more points on the curve. As a general rule the photographs taken on any one night were all on one plate, so that possible errors due to different treatment in development would be eliminated. When two plates were required they were developed together in one dish. Table I gives a list of the negatives obtained, with the dates and times of mid-exposure (expressed in fractions

* *Pub. Dom. Obs. Ottawa*, **9**, 23, 1925.
† *Ap. J.*, **32**, 302, 1910.
‡ *Ap. J.*, 64, 14 and 54, 1926.
§ *Pub. Dom. Obs. Ottawa*, **5**, 340, 1922.
‖ *Pub. Dom. Ap. Obs.*, **5**, 43, 1931.
¶ *Pub. A.S.P.*, **38**, 95, 1926.

of the Julian Day), also the phase in the intensity curve, computed as described later.

TABLE I

Neg.	Date 1933	J.D. 2427000 +	Phase	Neg.	Date 1933	J.D. 2427000 +	Phase
			d				d
4736	Mar. 30	162·419	0·123	4757	Apr. 13	176·438	0·016
4737	,, 30	·442	·008	4758	,, 13	·451	·029
4738	,, 30	·463	·029	4759	,, 13	·465	·043
4739	,, 30	·485	·051	4760	,, 13	·479	·057
4740	,, 31	·505	·071	4761	,, 13	·493	·071
4742	Apr. 1	164·384	·011	4762	,, 14	·507	·085
4743	,, 1	·403	·030	4763	,, 14	·521	·099
4744	,, 1	·423	·050	4764	,, 14	·535	·113
4745	,, 1	·443	·070	4765	,, 14	·549	·127
4746	,, 1	·463	·090	4766	,, 14	·563	·003
4747	,, 1	·483	·110	4767	,, 15	178·368	·007
4748	,, 2	·504	·131	4768	,, 15	·382	·021
4749	,, 2	·524	·013	4769	,, 15	·396	·035
4750	,, 2	·544	·033	4770	,, 15	·410	·049
4752	,, 13	176·368	·084	4771	,, 15	·424	·063
4753	,, 13	·382	·098	4772	,, 15	·438	·077
4754	,, 13	·396	·112	4773	,, 15	·451	·090
4755	,, 13	·410	·126	4774	,, 15	·465	·104
4756	,, 13	·424	·136	4775	,, 15	·479	·118
4757	,, 13	·438	·016	4776	,, 15	·493	·132

The changes observed in the spectra are sufficiently obvious to be noticeable without actual measurement. In general terms the spectral type varies from B3 to B4 or B5 (using the criteria adopted at this observatory for such classification).* This is specially noticeable in the group of lines at λ 4121 (*He*), λ 4128–31 (Si^+) and λ 4144 (*He*), in which the Si^+ double is sometimes fainter than λ 4121 and sometimes of the same intensity or even slightly stronger; also in λ 4472 (*He*) and λ 4481 (Mg^+), in which the relative intensities vary considerably. The individual lines which change in intensity most strongly are the *He* lines at λ 4388, 4144; the Mg^+ line at λ 4481; the Si^+ double at λ 4128–31; and the *C* line at λ 4267. The *He* lines 4121, 4009, and some other lines also vary in intensity though not to the same extent, while the *H* lines and the *He* lines at λ 4472, 4026 appear to be constant within the errors of measurement.

Commencing at zero phase (which represents a minimum intensity of *He* and a maximum of Mg^+ and Si^+) the following changes are observed: There is first a very rapid increase in the intensity of the *He* lines and 4267, accompanied by a decrease in the intensity of 4481 and 4128–31. The whole change takes place in slightly more than one hour, and appears to be continuous without any pauses or "humps" in the curve. This is followed immediately by a slower change in the reverse direction, the

* *M.N.*, **87**, 367, 1927.

maximum (of *He* intensities) being fairly sharply defined and the decline to a minimum taking about $2\frac{1}{4}$ hours. In some cases there is a slight suggestion of a "hump" in this downward part of the curve, but this is quite possibly spurious. If really present it is not prominent, and is usually masked by errors of measurement. Changes also occur in line

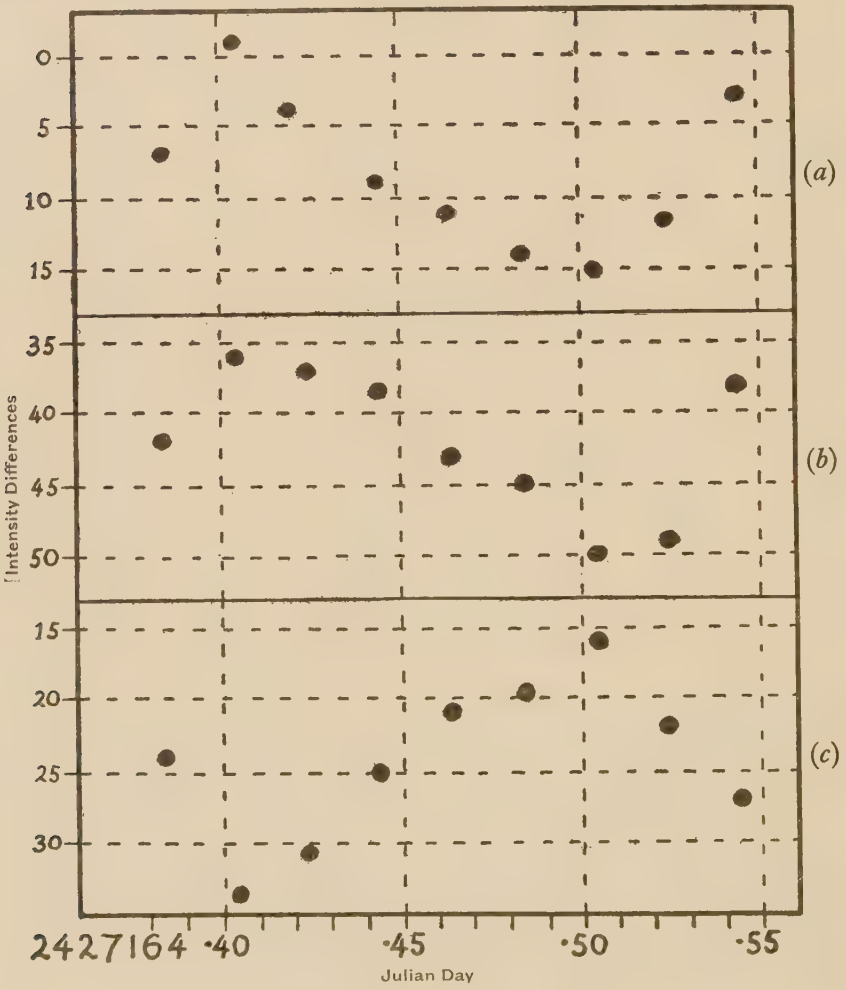


FIG. 1.—*Intensity Variations in the Spectrum of 42 Cam.*

(*a*) λ 4472–λ 4388; (*b*) *Hδ*–λ 4144; (*c*) λ 4472–λ 4481.

widths and sharpness. These, however, are more susceptible to differences of exposure or variations in atmospheric conditions; they are also more difficult to measure with the same accuracy as intensities. They have therefore not been studied systematically, but appear in general to be sharpest during the brief period of rise from minimum to maximum.

The intensities were all measured with the photometric wedge designed originally by Dr. W. J. S. Lockyer for spectroscopic parallax work. Only relative intensities were used, each variable line being compared with the

nearest line of constant intensity (such as $H\gamma$, $H\delta$, λ 4472 or λ 4026) and the difference of intensity of the two lines taken, thus minimising errors which might be introduced by variations in density in the different spectra. At least 10 settings were made on each individual line at one sitting, and

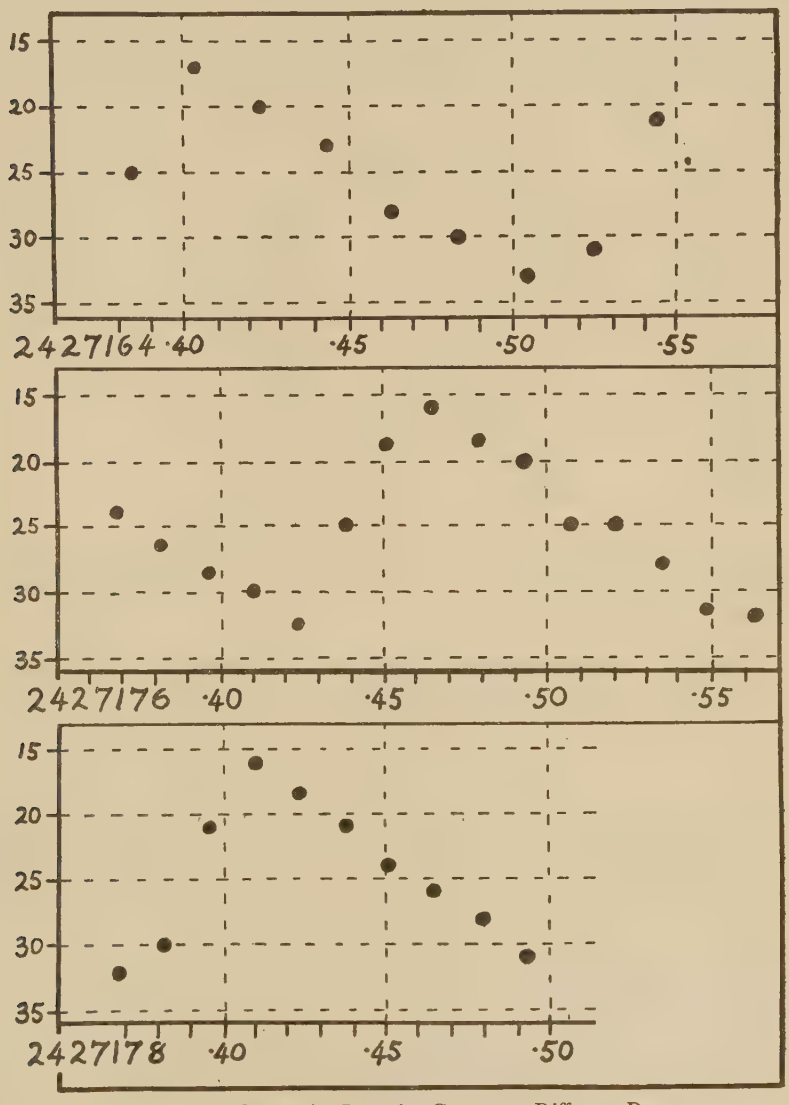


FIG. 2.—*Composite Intensity Curves on Different Dates.*

at least 3 sets of measures were made on different dates for each pair. The final mean values of the intensity differences thus represent the result of about 60–100 wedge settings for every point on the curve. The pairs of lines which are most easily measured are 4472–4481, 4472–4388 and $H\delta - 4144$. Results for these three pairs on the night of 1933 April 1–2 are shown in fig. 1. It should be noted that the intensity difference (as

measured) is small when the actual intensity of the variable line is great; the intensity scale is therefore made to increase downwards, so that a maximum in the curve represents a maximum *intensity* of the variable line (actually a minimum value of the measured difference). These diagrams illustrate sufficiently well the general remarks already made regarding the sequence of changes—particularly the sharp rise to a maximum (in the case of the *He* lines) with a slower fall, and the exactly reversed effect in the case of λ 4481 Mg^{+}. In fig. 2 are plotted the "composite intensity differences" obtained from all the measured lines. Such mean values have, of course, no special significance in themselves, but the process is useful for defining more precisely the shape of the curve, and especially in giving the epochs of maxima or minima with greater accuracy, since errors of observation tend to be smoothed out. Curves are given for three separate series of observations, showing how the shape is repeated at different dates.

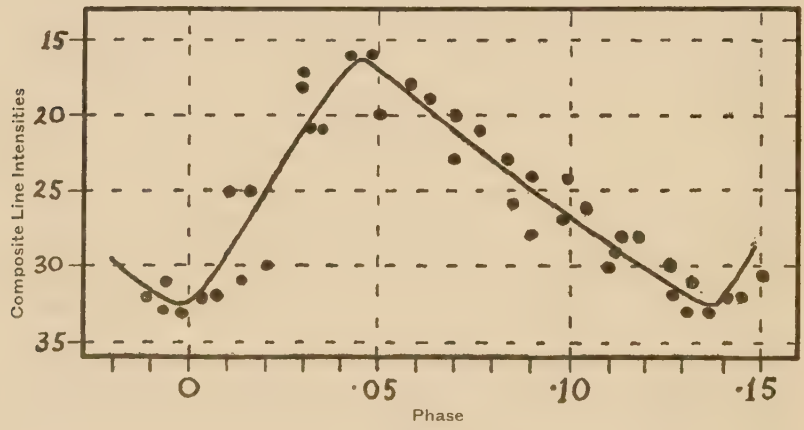


Fig. 3.—*Mean Intensity Curve.*

The period of variation, if estimated as well as possible from individual curves, appears to be about $0^{d}\cdot150$ on April 1, $0^{d}\cdot139$ on April 13 and $0^{d}\cdot136$ on April 15. This suggests a continuous decrease of the period during the time covered by the observations; but the differences are only slightly larger than the errors of observation, and may be purely accidental. In view of the fact that other stars of this type appear to show changes of line width of constant period, the evidence is insufficient to establish a variable period in this case, though its possibility must be considered. If a constant period is assumed, the value which best fits all the observations is $0^{d}\cdot1385$. Using this value, and an observed epoch of minimum at J.D. 2427164·511 as starting-point, the phases given in Table I were computed for each photograph. From these the mean curve shown in fig. 3 was obtained. Points representing the individual observations are introduced in order to show the amount of scatter, which may be due in part to a change of period (as suggested above), or a change of amplitude, or both, in addition to accidental errors. With regard to a possible variation in amplitude, the evidence is even less conclusive than that for the period, and suggests that

it was at least approximately constant during the period of investigation. From the mean curve the values of the following intervals were derived :—

Minimum to maximum = $0^d.043$.
Maximum to minimum = $0^d.095$.

The above elements were used to compute dates of maxima and minima corresponding to the observed dates, both of which are given in Table II.

TABLE II

Dates of Maxima and Minima

Maxima		Minima	
Observed	Computed	Observed	Computed
J.D. 2427000 +		J.D. 2427000 +	
164·407	164·416	162·44 ::	162·434
176·462	176·464	164·511	164·511
178·406	178·406	176·420	176·422
		176·559 :	176·560
		178·364 :	178·361
		178·500	178·499

The signs : and :: are used to indicate degrees of uncertainty in the observed epochs due to lack of points in the immediate neighbourhood, or from other causes.

Summary

Short-period variables of the β Canis Majoris type are frequently found to show variations in intensity of the spectral lines of a kind suitable for investigation with the equipment available at Sidmouth.

The star 42 Camelopardalis was photographed at regular intervals to ascertain if it belonged to this group of variables, and if so the nature of the spectrum changes which occur.

Definite intensity changes were found for the lines at $\lambda\lambda$ 4388, 4144, 4481, 4128–31, 4267, and (to a lesser extent) 4121, 4009. These were measured with a photometric wedge and are described in detail.

The period was found to be $0^d.1385$, with some evidence suggesting a possible variation. The rise to a maximum in the curve of intensities is very rapid, requiring only $0^d.043$, while the fall back to a minimum takes $0^d.095$. The shape of the curve thus bears some resemblance to the light curves of Cepheid variables.

THE EQUILIBRIUM THEORY OF THE ABUNDANCE OF THE ELEMENTS: A STATISTICAL INVESTIGATION OF ASSEMBLIES IN EQUILIBRIUM IN WHICH TRANSMUTATIONS OCCUR.

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(Communicated by C. H. Payne)

Introduction. The Astrophysical Problem

One of the most important problems requiring solution at the present time is that presented by the radiation of energy by the stars. It is generally agreed that the principal, if not almost the entire, source of this energy must be subatomic; beyond this there is as yet no considerable agreement. The possible sources of subatomic energy, as far as one can tell at the present time, are included among the following:—

1. Energy liberated during the transmutation of the elements.
2. The conversion of matter into energy by the "annihilation" of nuclei, nuclear constituents, or electrons.

Processes of the first sort are known to occur, since they have been observed in the laboratory. Natural transmutations are well known in the case of the radioactive elements, and are accompanied by a large liberation of energy; while artificial disintegrations caused by α particles,* giving rise to the emission of protons, with great energy in the case of aluminium, are likewise well known. Disintegrations produced by fast protons have been observed by Cockroft and Walton † among others, and in the case of lithium there is associated with its disintegration the liberation of a large amount of energy. There have also been observed transmutations ‡ in nuclei bombarded by fast α particles, as the result of which neutrons are emitted with a change of energy, and these same neutrons § are capable of producing further transmutations when they strike the nuclei of such elements as nitrogen. In fact, the nuclear reaction between α particles and the boron isotope of mass 11, resulting in the production of nitrogen and the liberation of neutrons, appears from experiments performed in the Cavendish Laboratory to be reversible:

$$B_{11}{}^{5} + (\alpha)_4{}^{2} \rightleftarrows N_{14}{}^{7} + (n)_1{}^{0}.$$

We do not propose to discuss this matter of the transmutations further in this place; from what has already been said it is obvious that the possi-

* Chadwick, Constable and Pollard, *Proc. Roy. Soc.*, A, **130**, 463, 1931.
† *Proc. Roy. Soc.*, A, **137**, 229, 1932.
‡ J. Chadwick, *Proc. Roy. Soc.*, A, **136**, 692, 1932.
§ N. Feather, *Proc. Roy. Soc.*, A, **136**, 709, 1932.

bility of transmutations of atomic nuclei has been established beyond any reasonable doubt by strong experimental evidence, and we may expect further that in most cases these transmutations will be accompanied by considerable absorptions or liberations of subatomic energy.

But with processes of the second sort, involving the "annihilation" of matter, things are on a different footing. With a single exception which will be considered later, and which probably has no immediate bearing on the liberation of energy in the stars, there does not appear to be any direct experimental evidence to show that the "annihilation" of matter can ever take place; and not only are there no experiments to show that particles can ever be "annihilated," but the thermodynamic and astrophysical consequences of supposing that particles can be "annihilated," and converted entirely into energy, are of quite an unsatisfactory nature. Mathematical investigations of the nature of this conversion, if it can occur, have been carried out by many people, including O. Stern,* E. A. Milne† and the author ‡—it appears that at terrestrial temperatures, or even at temperatures of millions of degrees, there would be no matter left if the conversion were free to seek its equilibrium state. It would be only at enormous temperatures, higher than 10^{11} degrees, that appreciable amounts of matter other than electrons could exist permanently in equilibrium with radiation; at lower temperatures all would be radiation. And if one postulates a critical temperature T_0 below which matter cannot seek its equilibrium with radiation, and above which it can, matter would be explosive—either no conversion would ever take place, or else if it ever started in a certain place where the temperature had reached the value T_0, the temperature would rise still more, and there would be an explosion with practically no matter remaining afterwards. Any theoretical investigation of the conversion of matter into radiation by the "annihilation" of particles must be highly speculative at the present time.

If the energies of stellar matter can be calculated for two different states C_1 and C_2, then the energy liberated when the matter changes from state C_1 to state C_2 is at once known; but if transmutations can occur then these energies are intimately related to the abundances of the elements in the stellar matter in the respective states. If we consider the problem of how energy is liberated by the transmutation of the elements, we therefore come up at once against the problem of the abundance of the elements. It will appear that we can answer at the present time this question: "What will be the abundances of the nuclei of the various elements in an assembly in statistical equilibrium § at a temperature T—an assembly containing radiation, electrons, protons, neutrons and atomic nuclei which are undergoing

* O. Stern, *Trans. Faraday Soc.*, **21**, 477, 1926.

† Unpublished. I am informed of this through the kindness of Professor Milne. See also Milne, *Nature*, **127**, 16, 1931. ‡ Unpublished.

§ By "statistical equilibrium" we of course refer to the state in which there are as many particles of kind s being formed per unit time as are being disintegrated per unit time. It is not necessary to assume that the equilibrium is possible—if the possibility of the reversal of transmutations is granted, then the possibility of equilibrium can be proved to exist.

all sorts of transmutations, without any 'annihilation'?" In answering this question we shall incidentally find out what the energies are, as well as the abundances, and we shall thus be in a position to predict the way in which energy must be liberated by the assembly when it is slowly cooled or compressed. If the interiors of the stars were completely insulated by being enclosed in perfectly non-conducting walls, then we might expect that the solution of this problem would apply to the stars directly. It appears in fact that the interior regions of some stars may be nearly perfectly insulated assemblies, and that our studies may well be applicable to them. While we have come upon this problem of the abundances in looking for a source of stellar energy, the problem itself is a fairly straightforward one in statistical mechanics, and as such we propose to investigate it in the following pages.*

§ 1. *The Statistical Problem*

Before we can proceed to the investigation of the assembly mentioned in the last paragraph, we must define more carefully the meaning of the expression, "without annihilation." While it is obvious that the meaning to be attached is that no ultimate nuclear constituent can ever be annihilated or created in our assembly—our assembly is to be one in which the total number of each sort of ultimate nuclear constituent, free and combined, is to remain constant—it is perhaps not quite so obvious what we should take these ultimate nuclear constituents to be. Until recently it was generally supposed that all nuclei could be regarded as being made of electrons and protons in various states of combination, and that in a nucleus of mass number M_s and atomic number N_s there were $M_s - N_s$ electrons and M_s protons. By the discovery of the neutron, however, the situation has become somewhat altered. It has been noticed for some time that the electrons in a nucleus do not appear to contribute to the spin or to the symmetry properties of the nucleus as a whole; and in fact the properties of nuclei, with the exception of those which exhibit β radioactivity, can best be explained by supposing that there are no electrons in the nuclei at all in any ordinary sense. On this view a nucleus should be looked upon as being made of $M_s - N_s$ neutrons and N_s protons. There is no experimental evidence to show whether or not a neutron is made of an electron and a proton. If it is made of † an electron and a proton, then we are still justified in considering the electron and the proton to be the two ultimate sorts of particle out of which all nuclei are made, and this hypothesis we shall denote by (a). But if the neutron is an ultimate particle which cannot ever be disintegrated into a

* Milne has already suggested the existence of equilibrium among electrons, protons and neutrons; and in fact has already suggested a law of "generation" of energy of the form $\epsilon = F(dT/dt)$; *Zeits. für Astrophysik*, 5, 337, 1932. While the combination of electrons and protons to form neutrons would furnish but little energy, the theory of the present paper will lead to a law of liberation of energy somewhat similar to the one proposed by Milne, while of course yielding considerably more energy.

† *I.e.* "If it can ever be disintegrated into . . ."

proton and an electron, then a reasonable procedure is to ignore for the present the difficult problem of β radioactivity and to suppose that all nuclei are composed of protons and neutrons. We shall call this latter hypothesis (*b*). To sum up this discussion, we propose to consider an assembly in equilibrium at a temperature T, in the two different fashions appropriate to the two different assumptions, respectively:

(*a*) The ultimate fundamental particles are electrons and protons, the total number of each of which remains constant in the assembly. A nucleus of mass number M_s and atomic number N_s contains $M_s - N_s$ electrons and M_s protons.

(*b*) The ultimate fundamental particles are electrons, protons and neutrons; which are all distinct and independent. The total number of each of these three kinds of particles remains constant in the assembly. A nucleus of mass number M_s and atomic number N_s contains $M_s - N_s$ neutrons and N_s protons.

It will appear that the development of the statistical theory will be nearly the same for both cases, (*a*) and (*b*); in fact, if a simple distinction is borne in mind, the same treatment will suffice for both.

There is one last matter to be mentioned before we pass on to the actual development of the theory. Recently, during experiments performed with expansion chambers by Anderson in California * and by Blackett and Occhialini † in the Cavendish Laboratory, tracks have been photographed which indicate that under certain circumstances, as under the influence of penetrating radiation, positively charged particles may be liberated which have masses considerably smaller than the mass of a proton, and which very possibly may turn out to have the mass of the electron. It is as yet too early to know the place which these particles will take in our scheme of things. If one could identify ‡ these particles with those predicted by Dirac some years ago, then one would be safe in supposing that they are not constituents of ordinary nuclei. Dirac § predicted the existence of "holes" in the distribution of electrons of negative energy. The holes would behave like particles of positive charge but would otherwise resemble electrons. The holes could presumably be created by the knocking of electrons out of states of negative energy into states of positive energy; the calculations of Dirac showed that these holes would soon be filled || again, after being created, and it followed that they could not play a rôle in the constitution of any matter which was at all permanent. It was observed in the experiments above referred to that electrons were liberated along with the positive particles. The possible creation of Dirac's particles

* *Science*, **76**, 238, 1932; *Phys. Rev.*, **43**, 491, 1933.

† *Proc. Roy. Soc.*, A, **139**, 699, 1933.

‡ This identification appears to have been made tentatively by Blackett and Occhialini, *loc. cit.*

§ *Proc. Roy. Soc.*, A, **126**, 360, 1929; *Proc. Camb. Phil. Soc.*, **26**, 361, 1930.

|| The possible occurrence of this annihilation is the exception referred to in the *Introduction*, and is not really "annihilation" in the sense in which the term has been used by astrophysicists in the past.

would not invalidate the proposed treatment of our assembly—the only effect would be to increase slightly the average number of free electrons, and our treatment would be correct so long as we used the true value of this quantity. If the recently discovered particles should turn out as the result of future experiments to be constituents of nuclei, our theory would of course have to be modified accordingly ; but the only course open to us seems to be to ignore for the present at any rate * these interesting particles, by accepting as plausible their identification with Dirac's "holes."

§ 2. *The Solution of the Statistical Problem*

Case (*a*).—Let us proceed first on the basis that assumption (*a*) is true —that electrons and protons are the ultimate particles, whose total numbers remain constant in the assembly despite all the various transmutations which can occur. Let the constant total number of electrons in the assembly be D_{el} and the constant total number of protons † be D_P. Let the gravitational potential be ϕ (we take the gravitational potential to be zero at infinity and to be positive in the vicinity of matter) and the electrostatic potential be W. We shall suppose that the assembly is of so small a volume that these quantities are sensibly constant throughout it; these conditions will certainly be realised in the interiors of stars, in volumes even as great as several cubic metres or more. The electrostatic and gravitational potentials may be expected to vary rapidly from particle to particle, but we may think of the actual potential at any point as being due to the superposition of a fluctuating field caused by the presence of charged, massive particles upon a steady field, nearly constant from place to place in the assembly, caused by the matter outside of it. Our ϕ and W refer to the latter slowly varying field. We take as the state of zero energy of a nucleus the state in which it is when it is completely dissociated into its constituent electrons and protons at rest in fields ϕ and W. With this convention, the energy of formation of a nucleus s of mass number M_s and atomic number N_s—that is, its energy when it is at rest in fields ϕ and W, is

$$Q_s = [c^2 - \phi][m_s - (M_s - N_s)m_{el} - M_s m_P], \qquad (2.0)$$

where c is the velocity of light, m_s is the mass of the nucleus, m_{el} is the mass of the electron, and m_P is the mass of the proton. The electrostatic

* Since this paper was written, the plausibility of the identification appears to have increased (see Oppenheimer and Plesset, *Phys. Rev.*, 44, 53, 1933), although pending a precise determination of the mass and charge the identification is not yet certain. I have calculated by statistical mechanics the increase in the number of free electrons (and in ξ'), caused by the creation of pairs of positive and ordinary electrons. The effect would be small except at very high temperatures, but to give the equations now would perhaps be premature.

An hypothesis that has been discussed somewhat, in the very latest literature, is that neutrons and electrons may be the ultimate particles, with protons and nuclei composed of neutrons and Dirac "holes." I have investigated this hypothesis by the usual statistical methods ; it proves to be really tantamount to our hypothesis (*a*), and requires no special consideration.

† For matter which is electrically neutral $D_{el} = D_P$ for both cases, (*a*) and (*b*).

potential W does not appear in equation (2.0) because charge is conserved when a nucleus is built up out of its constituents, and we shall therefore ignore it henceforth. The gravitational potential ϕ does appear, but it should almost always be possible to ignore it in comparison with c^2. The expression of Q_s in terms of the packing fraction $\mathbf{f}_H$ of hydrogen *atoms* and of the packing fraction $\mathbf{f}_s$ of *atoms* of kind s may be noted here: it is

$$Q_s = \frac{c^2 - \phi}{10{,}000\, N} M_s(\mathbf{f}_s - \mathbf{f}_H), \tag{2.01}$$

where N is the number of atoms in a gram atom.

If we use ${}_M A^N$ as the chemical symbol for a nucleus of mass number M and atomic number N then all possible transmutations may be regarded as being contained in the set of equations

$${}_M A^N \rightleftarrows (M-N) \text{ electrons} + M \text{ protons}$$

for all possible choices of M and N corresponding to real nuclei. The equilibrium state of the assembly is independent of the precise ways in which the reactions occur, and can be considered by the methods of § 21.4 of Fowler's *Statistical Mechanics*.*

There exist three independent parameters, ξ, η and θ $(=e^{-1/kT})$, corresponding to the thermodynamical partial potentials and to the absolute temperature respectively. The symbol k denotes Boltzmann's constant. By the usual methods we find for the mean value $\bar{X}_s$ of the number of particles of kind s in the assembly

$$\bar{X}_{el} = \xi \frac{\partial}{\partial \xi} \sum_{el\ r} \log(1 + \xi\theta^{\epsilon_r'}), \tag{2.02}$$

$$\bar{X}_P = \eta \frac{\partial}{\partial \eta} \sum_{P\ r} \log(1 + \eta\theta^{\epsilon_r'}), \tag{2.03}$$

and

$$\bar{X}_s = \xi^{M_s - N_s}\eta^{M_s} \frac{\partial}{\partial(\xi^{M_s - N_s}\eta^{M_s})} \sum_{s\ r} \log f(\xi^{M_s - N_s}\eta^{M_s}\theta^{\epsilon_r'}). \tag{2.04}$$

Here X_s is the number of nuclei (including neutrons) of kind s in the assembly, X_{el} is the number of free electrons (*i.e.* electrons which are not inside of nuclei) and X_P is the number of free protons. The sum $\sum_{el\ r}$ is taken over all possible energy levels ϵ_r' of an electron in accordance with the previously mentioned convention as to the state of zero energy; there are to be as many terms in $\sum$, corresponding to any degenerate level, as there are linearly independent wave functions to represent it. The sum $\sum_{P\ r}$ is a similar sum taken over all possible energy levels † of the proton,

* That the equilibrium state exists follows from Fowler, *loc. cit.*, § 5.5.

† In order not to encumber these expressions with more subscripts, we have used the same symbols ϵ_r' for the energy levels of particles of different kinds; the ϵ_r''s for a particle of one kind will of course be different in general from the ϵ_r''s of a particle of any other kind.

while $\sum_{s\ r}$ is the sum, for a particle of kind s, of the terms $\log f(\xi^{M_s-N_s}\eta^{M_s}\theta\epsilon_r')$ taken over all the energy levels ϵ_r' of a particle of this kind, with the usual conventions as to energies and degenerate levels. For a particle obeying the Einstein-Bose statistics (apparently the particles of this kind have even mass numbers), $f(q)=1/(1-q)$, while for a particle obeying the Fermi-Dirac statistics * (apparently the particles of this kind are the electron, proton, neutron and nuclei with odd mass numbers), $f(q)=1+q$.

Let us now define for each kind of particle s, other than electrons and protons, a parameter μ_s by the equation

$$\mu_s=\xi^{M_s-N_s}\eta^{M_s}e^{-Q_s/kT}. \tag{2.05}$$

We notice that any energy level ϵ_r' of a particle of kind s is equal to $Q_s+\epsilon_r$, where ϵ_r does not include the energy of formation of the nucleus out of electrons and protons. That is, when we use unaccented ϵ_r's, we are taking the state of zero energy † of the particle to be the state in which it is when it is at rest in a field ϕ. Then (2.04) becomes

$$\bar{X}_s=\mu_s\frac{\partial Z_s}{\partial\mu_s}, \tag{2.051}$$

where

$$Z_s=\sum_{s\ r}\log f(\mu_s e^{-\epsilon_r/kT}). \tag{2.06}$$

The other mean values can be obtained in similar fashion ; and for completeness we shall list them all here. The mean number of free electrons is

$$\bar{X}_{el}=\xi\frac{\partial Z_{el}}{\partial\xi}, \tag{2.07}$$

where

$$Z_{el}=\sum_{el\ r}\log(1+\xi e^{-\epsilon_r/kT}), \tag{2.08}$$

and the number of free protons is

$$\bar{X}_P=\eta\frac{\partial Z_p}{\partial\eta}, \tag{2.09}$$

where

$$Z_P=\sum_{P\ r}(1+\eta e^{-\epsilon_r/kT}). \tag{2.10}$$

The mean total energy $\bar{E}_{el}$ of all the free electrons is

$$\bar{E}_{el}=kT^2\frac{\partial Z_{el}}{\partial T}, \tag{2.11}$$

* Studies of the band spectra of homonuclear molecules force one to conclude that nuclei of even mass number obey the Einstein-Bose statistics, and that nuclei of odd mass number obey the Fermi-Dirac statistics. See, for instance, Mullikan, *Rev. of Modern Phys.*, **3**, 149, 1931.

† It is obvious that for the electron and proton the ϵ''s and ϵ's are equal.

the mean total energy $\bar{E}_P$ of all the free protons is

$$\bar{E}_P = kT^2 \frac{\partial Z_P}{\partial T}, \tag{2.12}$$

and the mean total energy $\bar{E}_s$ of all the particles of kind s, including the energy of formation, is

$$\bar{E}_s = kT^2 \frac{\partial Z_s}{\partial T}, \tag{2.13}$$

where the partial differential operator operates * on the μ_s as well as on the exponentials in Z_s. From (2.13), (2.05) and (2.051) we can derive the alternative expression

$$\bar{E}_s = \bar{X}_s Q_s + kT^2 \frac{\partial Z_s}{\partial T}, \tag{2.131}$$

where now the partial differential operator does *not* operate on μ_s. The first term, $\bar{X}_s Q_s$, is the mean total potential energy of formation of the particles of sort s. The second term includes the kinetic energy of the particles, their energy of nuclear excitation, and such potential energy as they may have from their presence in local fields of force.

The contribution p_{el} of the electrons to the total pressure is

$$p_{el} = kT \frac{\partial Z_{el}}{\partial V}, \tag{2.14}$$

where V is the volume of the assembly. The contribution p_P of the protons to the total pressure is

$$p_P = kT \frac{\partial Z_P}{\partial V}, \tag{2.15}$$

while that, p_s, of the nuclei s is

$$p_s = kT \frac{\partial Z_s}{\partial T}. \tag{2.16}$$

The mean number of electrons, ${}_{el}\bar{a}_r$, in the energy level ϵ_r is

$${}_{el}\bar{a}_r = \omega_r \xi \frac{\partial}{\partial \xi} \log (1 + \xi e^{-\epsilon_r/kT}), \tag{2.17}$$

the mean number of protons, ${}_P\bar{a}_r$, in the energy level ϵ_r is

$${}_P\bar{a}_r = \omega_r \eta \frac{\partial}{\partial \eta} \log (1 + \eta e^{-\epsilon_r/kT}), \tag{2.18}$$

while the mean number of particles s, ${}_s\bar{a}_r$, in the energy level ϵ_r is

$${}_s\bar{a}_r = \omega_r \mu_s \frac{\partial}{\partial \mu_s} \log f(\mu_s e^{-\epsilon_r/kT}). \tag{2.19}$$

In the above equations (2.17), (2.18) and (2.19) the symbol ω_r denotes the multiplicity of the energy level ϵ_r.

* It is well to bear in mind that the independent variables are ξ, η and T; μ_s is a function of these.

For the radiation in the assembly we obtain the usual laws of Stefan and of Planck :

$$\bar{E}_r = \frac{8\pi^5 k^4}{15c^3h^3} VT^4 \tag{2.20}$$

and

$$\bar{E}_\nu d\nu = \frac{8\pi h\nu^3 V d\nu}{c^3} \frac{1}{e^{h\nu/kT} - 1}. \tag{2.21}$$

Thus the statistical state of the assembly when it is in equilibrium is determined, and the important mean values can be worked out, if we know the values of T, ξ and η. The working out of the Z_s's will concern us in § 4, while the working out of the mean values will concern us in § 5. We note that the parameters ξ, η and T must of course be consistent with the equations

$$\begin{aligned} D_{el} &= \bar{X}_{el} + \Sigma_s \bar{X}_s (M_s - N_s), \\ D_P &= \bar{X}_P + \Sigma_s \bar{X}_s M_s, \\ E &= \bar{E}_r + \bar{E}_{el} + \bar{E}_P + \Sigma_s \bar{E}_s, \end{aligned}$$

and the parameters can in fact be considered to be determined by D_{el}, D_P and E. Here E is the total energy of the assembly.

Case (*b*).—Let us proceed next to develop the theory on the basis of assumption (*b*)—that electrons, protons and neutrons are independent ultimate particles ; that a nucleus of mass number M_s and atomic number N_s contains N_s protons and $M_s - N_s$ neutrons ; and that D_{el}, D_P and D_n are constant in the assembly. D_{el} and D_P are respectively the total numbers of electrons and of protons as before, while D_n is the total number of neutrons in the assembly. As before, the electrostatic field may be ignored ; and if we take as the state of zero energy of a nucleus the state in which it is when it is dissociated into protons and neutrons at rest in a gravitational field of potential ϕ, then the potential energy of a nucleus s when it is at rest in a field ϕ—that is, its energy of formation out of protons and neutrons—is given by

$$Q_s' = [c^2 - \phi][m_s - (M_s - N_s)m_n - N_s m_P], \tag{2.50}$$

where m_n is the mass of a neutron.

By the usual methods * we find that there exist four parameters, ξ', η, ζ and θ, corresponding to the thermodynamic partial potentials and to the temperature respectively. For each composite nucleus s we define a quantity μ_s by the equation

$$\mu_s = \eta^{N_s} \zeta^{M_s - N_s} e^{-Q_s'/kT}. \tag{2.52}$$

Then if we consider that for such a composite nucleus s any energy level ϵ_r' is equal to $Q_s' + \epsilon_r$, where ϵ_r is the energy measured from the state in which the nucleus s is at rest in a field of potential ϕ as in case (*a*), we find that the mean number of particles of kind s in the assembly is given by

* The generalization of Fowler's § 21·4 to include three kinds of ultimate particles is obvious.

$$\bar{X}_{el} = D_{el} = \xi' \frac{\partial}{\partial \xi'} \sum_r \log (1 + \xi' e^{-\epsilon_r/kT}), \tag{2.53}$$

$$\bar{X}_P = \eta \frac{\partial}{\partial \eta} \sum_r \log (1 + \eta e^{-\epsilon_r/kT}), \tag{2.54}$$

$$\bar{X}_n = \zeta \frac{\partial}{\partial \zeta} \sum_r \log (1 + \zeta e^{-\epsilon_r/kT}), \tag{2.55}$$

$$\bar{X}_s = \mu_s \frac{\partial}{\partial \mu_s} \sum_r \log f(\mu_s e^{-\epsilon_r/kT}). \tag{2.56}$$

Here $\bar{X}_n$ is the mean number of free neutrons (*i.e.* neutrons which are not inside of nuclei) in the assembly. Corresponding to equation (2.131) we have also

$$\bar{E}_s = \bar{X}_s Q_s' + kT^2 \frac{\partial}{\partial T} \sum_r \log f(\mu_s e^{-\epsilon_r/kT}). \tag{2.565}$$

Now define a parameter ξ by the equation

$$\zeta = \xi \eta e^{-\frac{c^2-\phi}{kT}(m_n - m_{el} - m_P)} \tag{2.57}$$

and replace ζ wherever it occurs in the preceding six equations by the right-hand member of equation (2.57). Then equation (2.52) becomes transformed into

$$\mu_s = \xi^{M_s - N_s} \eta^{M_s} e^{-\frac{c^2-\phi}{kT}[m_s - (M_s - N_s)m_{el} - M_s m_P]} \tag{2.58}$$

which is simply equation (2.05); while equations (2.53), (2.54), (2.56) and (2.565) are unaltered. (2.55) is transformed into

$$\bar{X}_n = \mu_n \frac{\partial}{\partial \mu_n} \sum_r \log (1 + \mu_n e^{-\epsilon_r/kT}), \tag{2.59}$$

where $\mu_n \equiv \zeta$ is defined by (2.57) or (2.05) in terms of ξ and η.

We find in similar fashion that all the equations obtained in case (*a*) for the mean values of the numbers of particles, for the pressures, and for the mean numbers of particles ${}_s\bar{a}_r$ in levels ϵ_r, hold in case (*b*) so long as we replace ξ, wherever it occurs in the equations describing the statistical properties of the *electrons*, by ξ'. The only equations to be thus altered are (2.02), (2.07), (2.08), (2.11), (2.14) and (2.17). The energies $\bar{E}_s$ are given by (2.131) with * Q_s replaced by Q_s'. In case (*b*) there are thus four

* If we like, we can even leave Q_s unchanged in (2.131), without replacing it by Q_s'. Failing to replace Q_s by Q_s' will give us an energy greater than it should be by an amount $\bar{X}_s(Q_s - Q_s') = \bar{X}_s(M_s - N_s)Q_n$, where $Q_n = (c^2 - \varphi)(m_n - m_{el} - m_P)$. If we follow this procedure we must use (2.131) for neutrons as well; and then the only effect is to add to the total energy of the assembly a constant $Q_n D_n$. This is tantamount merely to taking a slightly different convention for the measurement of energies, and so cannot introduce any errors. We are concerned really with *differences* of energy, always.

independent parameters, ξ', η, ξ and T. These four parameters must of course be consistent with the equations

$$\begin{aligned} D_{el} &= \bar{X}_{el}, \\ D_P &= \bar{X}_P + \Sigma_s \bar{X}_s N_s, \\ D_n &= \bar{X}_n + \Sigma_s \bar{X}_s (M_s - N_s), \\ E &= \bar{E}_r + \bar{E}_{el} + \bar{E}_P + \bar{E}_n + \Sigma_s \bar{E}_s, \end{aligned}$$

and can be considered to be defined by them. Whereas in case (*a*) the two parameters ξ and ξ' are identical, in case (*b*) they are different; and if this distinction is borne in mind, one general scheme will hold for the solutions of both cases, (*a*) and (*b*).

The Effects of the Presence in the Assembly of Two or More Different Phases of One or More Substances.—In the treatment of case (*a*) we stated that the three parameters ξ, η and T were independent; and in the treatment of case (*b*) we stated that the four parameters ξ', η, ξ and T were independent. These statements are strictly true only when there is no substance present in two or more different phases in the assembly. A simple illustration of the consequences of there being a substance present in the assembly in two different phases is provided by an assembly in which some substance s is present in both the gaseous and crystalline phases. If both crystals and gas exist in the assembly in equilibrium, μ_s becomes a function * merely of the temperature T. There is one less independent variable when the crystalline phase is present as well as the gaseous, than there is when the crystalline phase is absent and only the gaseous phase is present. This is true in general, whatever the nature of the phases. If two different phases of the same substance s are present, there is one less independent variable than there is if only one phase of the substance s is present. If three different phases of the same substance s are present, the temperature and μ_s are determined. There are thus two independent variables less. There can be in general no more than one substance present in three phases. Each time a new phase is incorporated into an assembly in which initially each substance is present in only one phase, the number of independent variables is reduced by unity. There can thus be present in general no more than three of these "extra" phases in case (*a*) and no more than four in case (*b*), if the assembly is in equilibrium.

§ 3. *A Recapitulation. General Observations about the Solution*

A Recapitulation of the Results.—All the important mean values can be obtained from the formulæ of the preceding section. Since the formulæ are somewhat scattered, however, it may be well to summarise briefly the general solution which holds for both cases (*a*) and (*b*), before proceeding further.

There exist three independent parameters: T, ξ and η for both cases,

* If $K_s(T)$ is the crystalline partition function "per molecule," then $\mu_s = [K_s(T)]^{-1}$. See Fowler, *Statistical Mechanics*, p. 117; or Fowler and Sterne, "Statistical Mechanics," *Rev. of Modern Physics*, 4, 650, 1932.

(*a*) and (*b*). There is a fourth parameter ξ', which for case (*a*) is identical with ξ, but which in case (*b*) is an independent parameter with the same status as ξ and η. For each nucleus s (including the neutron) of mass m_s, mass number M_s and atomic number N_s, there exists a parameter μ_s defined by the equation (2.05) where Q_s is defined by equation (2.0) or (2.01). The parameter μ_s for the electron is ξ', and that for the proton is η:

$$\mu_{el} = \xi',$$
$$\mu_P = \eta.$$

With this understanding, there exists for each kind of particle s (which may now be an electron, a proton, a neutron or any composite nucleus) a function Z_s of μ_s, T and V defined by equation (2.06), with the form for f which is appropriate to the kind of particle considered. The mean number $\bar{X}_s$ of particles of the kind s is then given by equation (2.051) in all cases; the mean total energy of these particles, $\bar{E}_s$, is given by equation (2.131); the contribution p_s of these particles to the total pressure is given by equation (2.16); while the mean number of particles s, ${}_s\bar{a}_r$, in the energy level ϵ_r is given by equation (2.19). There are, as well, the laws of Stefan and of Planck. Thus the statistical state of the assembly when it is in equilibrium is determined, and the important mean values can be worked out, if we know the values of T, ξ, η and possibly of ξ'. The calculation of the Z_s's will concern us in § 4, while the working out of the mean values will concern us in § 5.

General Observations about the Solution.—The mean values of the numbers of particles of the various kinds, $\bar{X}_s$, appear therefore as functions of the respective μ_s's and of T. In case (*b*), if we are not interested in the number of free electrons, we can ignore ξ'. The number $\bar{X}_s$ is always a monotonic increasing function of μ_s for any particular value of T, but there is a marked difference between the dependence of $\bar{X}_s$ upon μ_s in the case of "even" particles which obey the Einstein-Bose statistics on the one hand, and the dependence of $\bar{X}_s$ upon μ_s in the case of "odd" particles which obey the Fermi-Dirac statistics on the other. For the latter, as μ_s varies from 0 to plus infinity, $\bar{X}_s$ varies from 0 to plus infinity. For the former, as μ_s increases from 0, $\bar{X}_s$ increases at first in much the same way as for odd particles; but as μ_s approaches 1, $\bar{X}_s$ approaches plus infinity; and μ_s can never exceed unity for any "even" particle * when the assembly is in equilibrium. This introduces a severe limitation upon the possible values of ξ and η, as we shall see shortly.

From (2.05) we have

$$T \log \mu_s = (M_s - N_s) T \log \xi + M_s T \log \eta - Q_s/k, \qquad (3.06)$$

and if we ignore the trivial ϕ in (2.01) we can replace the last term in (3.06) by its expression in terms of the packing fraction of the complete *atom* s, obtaining

$$T \log \mu_s = (M_s - N_s) T \log \xi + M_s T \log \eta + \frac{c^2}{10{,}000\, R} M_s (\mathbf{f}_H - \mathbf{f}_s), \qquad (3.07)$$

* These statements are in anticipation of § 4.

where R is the gas constant per gram molecule, $\mathbf{f}_H$ is the packing fraction* of the hydrogen *atom*, and $\mathbf{f}_s$ is the packing fraction of the *atom* s. Let $x = T \log \xi$, $y = T \log \eta$. Consider the lines

$$0 = (M_s - N_s)x + M_s y + \frac{c^2}{10{,}000\, R} M_s(\mathbf{f}_H - \mathbf{f}_s)$$

for all the elements. We propose to call these lines "abundance" lines. We have from (3.07) that $-T \log \mu_s$ is equal to the product of M_s and the distance, measured parallel to the y axis and in the sense of increasing y, from the point (x, y) to the abundance line of the element s. The position

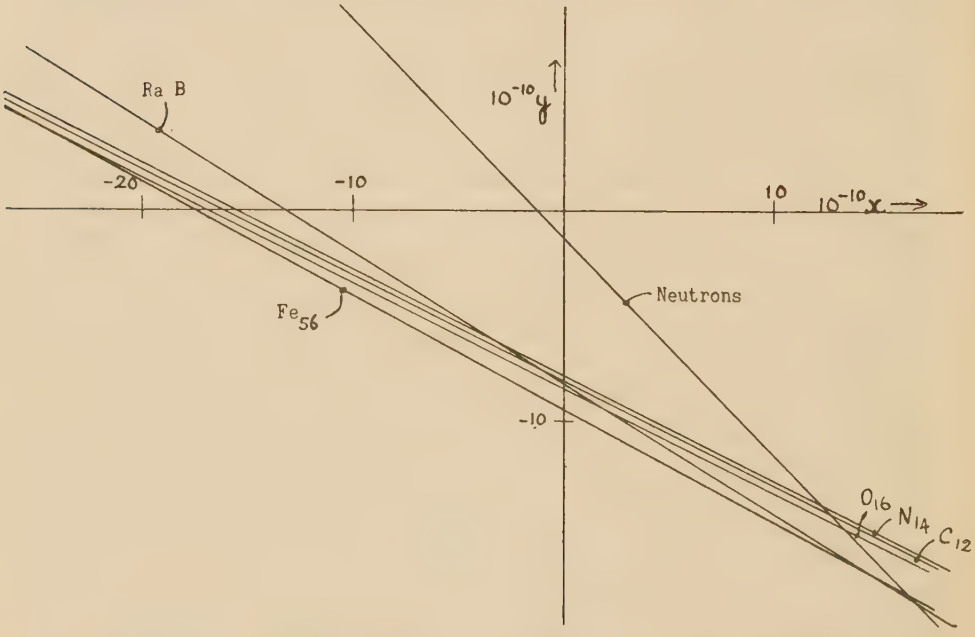

FIG. 1.—*The Abundance Lines of Neutrons*, N_{14}, C_{12}, O_{16}, Fe_{56} *and* Ra B_{214}.

of the point (x, y) in the x, y plane, together with the temperature T, determines the statistical state of the assembly in equilibrium.†

Consider next the abundance lines of the "even," Einstein-Bose, particles. In fig. 1 the lines have been drawn for neutrons, Fe_{56}, O_{16}, C_{12}, N_{14} and $Ra\,B_{214}$; all of these except neutrons are "even." The lines for iron and *Ra B* are included, although their packing fractions are not accurately known, for reasons which will appear shortly. The packing fraction of Fe_{56} is taken from the packing fraction curve ‡ to be -9; that of *Ra B* is taken by extrapolation from the curve to be 2. The packing fraction of O_{16} is 0, and the packing fractions of C_{12} and N_{14} are known to

* The packing fraction is the atomic mass, less the mass number, all multiplied by 10,000 and divided by the mass number.

† . . . except as regards the properties of electrons in case (*b*). To know these in case (*b*) we need to know ξ' as well as x, y and T.

‡ *Radiations from Radioactive Substances*, Rutherford, Chadwick and Ellis, p. 528.

be approximately 3 and 5·7 respectively.* The packing fraction of neutrons is taken † to be 66·5. It is at once obvious that points (x, y) which are possible for the assembly in equilibrium must lie below all the abundance lines for the even elements.‡ At most, the point (x, y) could lie on the lowermost line of all the even lines; in that case the corresponding element would be present in the assembly with an infinite density. In possible cases, the point (x, y) can come very close to the lowest even line and that element will then be present to the practical exclusion of all other elements —even though (x, y) may lie slightly above the lines for one or more odd elements. It is at once obvious from what has been said that we should expect to find a general preponderance of the elements of even atomic mass. A preponderance of this nature has long been observed in terrestrial and stellar abundances.§ It also appears that if any element is to be present in overwhelming preponderance at low temperatures, it must probably be associated with the lowest line in the set of even lines, so long as there is no odd line lying too far below it. Since the lines have various slopes, it should be noticed, however, that the line which is the lowest for a certain value of x need not be the lowest for a different value of x.

We cannot proceed much further here without making some assumption which will enable us to find the values of x which we may expect to concern us. If we adopt assumption (*a*) and so identify ξ with ξ', then x can be calculated || from the total density and the temperature. For temperatures subject merely to the restriction that $T << 1{\cdot}8 \times 10^5 \rho^{2/3}$, we find that $x = 1{\cdot}9 \times 10^5 \rho^{2/3}$ and thus as long as $\rho < 4 \times 10^5$, $x < 10^9$. Thus for these temperatures, and densities smaller than 4×10^5 grm./cm.^{3}, the interesting part of the abundance diagram lies very close to, and to the right of, the y axis; and in this region the lowest even abundance line is very probably the line corresponding to the even element with the smallest packing fraction, because that line has the lowest y intercept. In this region the corresponding element may be expected, in general, to be the most abundant element at low temperatures, in an assembly in equilibrium. The lowest point on the packing-fraction curve lies close to iron and nickel, and if Fe_{56} were actually the even element with the smallest packing fraction, then the abundance of iron in the stars, its great preponderance in many meteorites, and its possible preponderance inside of the Earth could all be understood. There is another conclusion of a simple nature to be drawn from fig. 1—at sufficiently

* Rutherford and others, *loc. cit.*, p. 527. That of H is taken from the table to be 77·8.

† J. Chadwick, *Proc. Roy. Soc.*, A, **136**, 692, 1932.

‡ We use the word "element" rather than "isotope" throughout this paper; it is perhaps needless to point out that for our purposes two isotopes of the same element are quite as distinct as two isotopes of different elements are. When we speak of "the element *s*" we refer to a particular isotope of a particular element. We use the one word "element" because the usual grouping together of certain isotopes according to their chemical properties is without importance to us.

§ With the notable exception that hydrogen, presumably of odd mass, is very abundant in most stellar atmospheres.

|| The reader is referred to § 5, ordinary degenerate case.

large densities, much larger than 10^6 grm./cm.^{3}, the line of greatest negative slope must eventually become the lowest line. It at once appears that the line for neutrons (which are odd) has by far the greatest negative slope of all the lines of the known elements, while the line of *Ra B* (which is even) has the greatest negative slope of all the even elements. At sufficiently enormous densities, for regions of the diagram for which the neutron line * lies far below all the other lines, the assembly at low temperatures should contain a preponderance of neutrons. This assertion is not particularly dependent upon the value of the packing fraction ; the slope of the abundance line is known in terms merely of the mass number and atomic number. At these high densities, matter at low temperatures would be literally squeezed together into the form of neutrons.

We close this section with the obvious observation that since the lighter elements of even atomic number as a rule have smaller † packing fractions than the lighter elements of odd atomic number, the μ's of the former elements will as a rule be larger than the μ's of the latter elements. There should therefore be a general preponderance of elements of even atomic number, among the lighter elements, for both cases, (*a*) and (*b*). A general preponderance of this nature has long been noticed in the lithosphere and in stellar atmospheres.‡

§ 4. *Calculation of the* Z_s*'s*

General Considerations.—We should expect the equilibrium in which we are interested to exist in the stars if it exists at all. On the surface of the Earth it certainly does not exist—we have only to look to the continued existence of lithium of mass number 7 to be convinced of this. Inside of the Earth, due to the somewhat greater density, there appears to be some possibility, though a remote one, that the equilibrium § could be established. The evidence appears to be that diffuse matter in space is in no condition

* For this to happen, the density should be greater than approximately $2{\cdot}3 \times 10^{10}$ grm./cm.^{3} when $T << 6 \times 10^7 \rho^{1/3}$.

† Rutherford and others, *loc. cit.*, pp. 529, 530.

‡ . . . with the notable exception that hydrogen, of atomic number 1, is very abundant in most stellar atmospheres.

§ Although the Earth is probably not in equilibrium as far as transmutations are concerned, one may perhaps consider it to contain *at least some* matter which came originally from an assembly in equilibrium—and so the task of explaining the constitution of the Earth is perhaps not an impossible one. At the same time the task is difficult. Considerations of density make it certain that the central regions of the Earth contain matter of quite different composition from that of the crust. See, for instance, Jeffreys, *The Earth.* Urey and Bradley studied the relative abundances of isotopes in the Earth's crust (*Phys. Rev.*, 38, 718, 1931), and concluded that the observed ratios were not consistent with equilibrium at any temperature. They used, however, the classical, and not the quantum, statistics ; while the abundances in the Earth's crust cannot be typical of the whole of the interior. The observed general preponderance of elements of even mass and number in the Earth's crust is if anything in agreement with the predictions of the quantum statistics, and because of the complexity of the problem this rough agreement is perhaps as much as one could hope for, at present, even if the matter of the Earth had once been in equilibrium. The problem deserves further treatment.

approximating to thermal equilibrium. The atmospheres of stars are certainly not in equilibrium.* The only other places where we may expect matter to be in equilibrium are the interiors of the stars ; and here the conditions appear to be quite favourable. Temperatures and pressures, and often densities, are high. Moreover, it is fairly certain that whatever changes of temperature occur in a region in the interior of a star occur very slowly,† and except for a slight anistropy in the radiation due to its having a net flux away from the centre of the star, matter is remarkably well insulated. Both the rate of loss of energy and the anisotropy are very small. Consider the case of the star V Puppis A. It is radiating energy at the rate of approximately 700 ergs/grm. sec., an exceptionally large value for a star. Its central temperature is probably greater than 42,000,000 °K.‡ If a cubic centimetre is considered in this star at a place where the temperature is 10^7 °K., the energy of the radiation alone in this volume is approximately 8×10^{13} ergs. Since the radius § of the star is $5{\cdot}28 \times 10^{11}$ cm., the average liberation of energy works out as nearly 40 ergs/cm.^{3} sec. At this rate the cubic centimetre enclosure is losing no *more* than 1 per cent. of its energy every 500 years, which indicates the perfection with which the enclosure is insulated by the material around it! If energy is liberated uniformly throughout the volume of the star, the flux of energy outwards through a square centimetre normal to the radius vector is greatest at the surface, where it corresponds in V Puppis A to an effective temperature of about 20,000°. In one second the energy passing outwards through a square centimetre normal to the radius vector, in our enclosure, can thus exceed the energy passing inwards by approximately 1 part in 6×10^{10} at most. The asymmetry of the radiation is thus trivial, and could not be expected to interfere to any extent with the nearly perfect establishment of equilibrium among the nuclei. If we are to attempt to apply our theory to actual matter, we should therefore seek to apply it to the interiors of stars ; and we should calculate the Z_s's with particular reference to the conditions of temperature and density there to be expected.

Most stellar theories agree in predicting high temperatures, great

* The lower the temperature is, the more slowly will transmutations occur ; and the more perfectly will matter have to be insulated to enable the equilibrium of transmutations to be established. The atmospheres of stars are at temperatures which appear to be considerably too low for there to be any stimulation of transmutations, and thus, since the matter is badly insulated as well, there appears to be no opportunity for the equilibrium to be established either in stellar atmospheres or indeed in the outer regions of a star. The observed preponderance of hydrogen in most stellar atmospheres is evidence for the truth of this, since if equilibrium were by any chance established at such low temperatures, practically no hydrogen could be expected to be present.

† We exclude, of course, variable stars and novæ from this discussion.

‡ Jeans, *Astronomy and Cosmogony*, 2nd ed., p. 129, gives the following values : $M = 19{\cdot}2 \times \odot$; $T_c = 160{,}000{,}000$; $L/M = 1100$. Eddington, *The Internal Constitution of the Stars*, p. 147, gives the values : $M = 19{\cdot}2 \times \odot$, $T_c = 42{,}400{,}000$; $L = 2{\cdot}62 \times 10^{37}$ ergs/sec. We take Eddington's value of L/M. If Milne's theory is correct, values of T_c can occur considerably greater even than $1{\cdot}6 \times 10^8$ °K.

§ Eddington, *loc. cit.*

densities, or both high temperatures and great densities for stellar material inside the star. In the case of high temperatures and low densities we shall suppose for simplicity that all the nuclei are completely ionized; so that the matter will be a mixture of bare nuclei and electrons. At the temperatures for which appreciable quantities of the very light or very heavy elements can exist in equilibrium this is not too unreasonable a supposition; even if some ion structures of the ions heavier than hydrogen remain they can hardly be important to us, because the energy required to ionize an atom *completely* is trivial in comparison with the energy of formation, Q_s or Q_s', of the nucleus. At very great densities, whatever the temperature may be, the notion of ionization loses its meaning and we must think of matter as an intimate mixture of nuclei and electrons.

It is easy to show that stellar matter must be electrically neutral on the average; indeed it can be shown that the difference between the mean total charges of the electrons and of the nuclei in a moderately large volume in a star cannot be greater numerically than approximately 10^{-35} times the mean total nuclear charge in the volume.* Small-scale fluctuations, however, may have an appreciable effect upon the equilibrium of the nuclei. It is a matter of regret that no satisfactory theory of the electrostatic effects appears to have been developed for stellar material. The theory of Debye and Hückel for the case of electrolytes is somewhat deficient for electrolytes,† and is still less correct when applied to stellar matter of great density—for it makes use of the classical statistics and of Boltzmann's theorem, and supposes that the mutual electrostatic energy of two particles close together is small in comparison with kT. It further takes no account of the quantization of the energies of those electrons which are close to nuclei. If we do use the theory, we make use of the value

$$\frac{\Psi_\epsilon}{k} = \frac{2\sqrt{\pi}\epsilon^3 V}{3(kT)^{3/2}}\left\{\frac{X_{el}}{V} + \sum_s N_s^2 \frac{X_s}{V}\right\}^{3/2}$$

which it predicts ‡ as the electrostatic contribution § to the characteristic function (divided by k) when all the nuclei are completely ionized. It can be shown from this that the electrostatic effects are trivial for matter made mostly of radium when the density

$$\rho << 20T,$$

for matter made mostly of iron when

$$\rho << 40T,$$

and for matter made mostly of oxygen when ‖

$$\rho << 100T.$$

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* See, for instance, Eddington, *loc. cit.*, § 191.

† Fowler, *Statistical Mechanics*, pp. 195, 322.

‡ Fowler, *loc. cit.*, p. 349, equation (1016).

§ ϵ is the charge of the electron.

‖ That is, the errors in the equilibrium values of X_s, calculated by the theory which ignores electrostatic effects altogether, are trivial for *all* particles when the *chief* constituents of the assembly are those stated, under the stated conditions of density and temperature.

The restrictions on the density, for the electrostatic effects to be trivial, appear to be decreasingly severe for the lighter * elements. Electrons obeying the Fermi-Dirac statistics have large kinetic energies at all temperatures, when the density is 1 or more. Since electrons are the most abundant particles, the "temperature" T to be used in the Debye-Hückel theory should be very high, and should be taken perhaps from the Sommerfeld theory of metallic conduction to be 10,000° or more, for matter with a density of 1, even at low temperatures. Some slight basis is thus provided, by the theory of Debye and Hückel, for the hope that when the electrostatic effects are eventually included in a correct theory they will perhaps prove to be small. Until then we are rather compelled to ignore them altogether.

The Calculation of the Z_s*'s.*—When we calculate the Z_s's we must include in the energies both kinetic energies and energies of nuclear excitation ; potential energy due to the presence of a particle s in a gradually varying field due to long-range forces is accounted for in the value of μ_s. There appear to be two main cases, each of which consists of two sub-cases. If μ_s is small compared with unity, we can use the classical approximations to the correct Einstein-Bose and Fermi-Dirac statistics. We call this case the "classical" case. If μ_s is not small compared with unity, we call the statistics "degenerate." In the classical case, we shall consider the two extreme sub-cases where (*a*) $kT << m_sc^2$ and (*b*) $kT >> m_sc^2$. In the former sub-case the ordinary results are obtained, while in the latter sub-case the relativistic change of mass becomes important and we call this sub-case the "relativistic" classical case. We shall not work out Z_s for intermediate values of T. Similarly, there are two sub-cases for the degenerate statistics —either the important energies in the sums are trivial compared with m_sc^2 or they are not. The former sub-case, for $kT \log \mu_s << m_sc^2$, we shall call the "ordinary" degenerate case ; the latter the "relativistically" degenerate case. We shall work out the latter case for $kT \log \mu_s >> m_sc^2$ only.† We can thus classify the cases which we shall discuss as follows :—

Case 1. Classical Statistics, $\mu_s << 1$.

1 (*a*). Ordinary Classical Statistics, $kT << m_sc^2$.
1 (*b*). Relativistic Classical Statistics, $kT >> m_sc^2$.

Case 2. Degenerate Statistics, $\mu_s \sim 1$ or $\mu_s > 1$.

2 (*a*). Ordinary Degenerate Statistics, $kT \log \mu_s << m_sc^2$.
2 (*b*). Relativistically Degenerate Statistics, $kT \log \mu_s >> m_sc^2$.

* From the expression for Ψ_ϵ in the text, the electrostatic effects would (in matter of given constitution) be important increasingly for heavier constituents, and would result in there being *greater* amounts of these heavy constituents than would be given by the theory which ignored the effects altogether. In view, however, of the uncertainty connected at present with the calculation of Ψ_ϵ, it is perhaps best not to present the analysis here. The limits given in the text are calculated for the constituent uranium, and thus hold *a fortiori* for lighter constituents, under the stated conditions.

† For the calculation of pressures and energies in the case of T nearly zero and $kT \log \mu_s$ comparable with m_sc^2, see Stoner, *M.N.*, **92**, 651, 1932. Stoner investigates electrons only, but no other particle is likely to fall under this case.

In both Cases 2 we shall have to distinguish between odd and even particles ; in both Cases 1 the odd and even particles behave similarly, so that it is not necessary to distinguish between them.

Case 1 (*a*)

$$(\mu_s << 1, \quad kT << m_s c^2)$$

If there are no nuclear energy levels other than the nuclear ground level, it follows from equation (1513) of Fowler's *Statistical Mechanics* that for both even and odd particles, very nearly,

$$Z_s = \mu_s \omega_0 \frac{(2\pi m_s kT)^{3/2}}{h^3} V, \tag{4.0}$$

where ω_0 is the statistical weight of the state of the nucleus, and is the number of linearly independent wave functions capable of representing the nucleus. So far as is known at present, $\omega_0 = 2I + 1$ where I is the value of the spin of the nucleus in units $h/2\pi$. If there are excited nuclear levels possible, then from (F 1514)* we have at once, very nearly,

$$Z_s = \mu_s \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum\nolimits_n \omega_n e^{-\epsilon_n/kT} \tag{4.01}$$

since the other terms are trivial. Here ω_n is the statistical weight of the nth nuclear level, and the ϵ_n's are the energies of the nuclear levels, taking the nuclear ground level to be of zero energy. For electrons, protons and neutrons so far as we know there are no excited levels, and the spins† are $\frac{1}{2}$, so that we should use for these particles equation (4·0) with $\omega_0 = 2$.

Case 1 (*b*)

$$(\mu_s << 1, \quad kT >> m_s c^2)$$

Consider first the case when there are no excited levels. The kinetic energy E of a particle whose rest mass is m_s is given by the equation

$$E = m_s c^2 \left[\left(1 + \frac{p^2}{m_s{}^2 c^2} \right)^{1/2} - 1 \right]. \tag{4.10}$$

Here p is the momentum of the particle. It follows accurately from equation (F 1507) that the possible values of the momentum of the particle, if it is confined in a rectangular box of sides a, b and c, are given by

$$p^2 = \frac{h^2}{4} \left(\frac{s'^2}{a^2} + \frac{t^2}{b^2} + \frac{u^2}{c^2} \right), \tag{4.11}$$

where s', t and u are any positive integers other than zero. There is only

* The number of an equation when it is preceded by "F" will refer to Fowler's *Statistical Mechanics*.

† This value for the spin of the neutron is perhaps somewhat uncertain.

one linearly independent wave function, apart from internal motions, for each choice of s', t, u. Let the statistical weight of the nucleus be ω_0. Then, very nearly,

$$Z_s = \omega_0 \mu_s \sum_{s'=1}^{\infty} \sum_{t=1} \sum_{u=1}^{\infty} e^{-E/kT},$$

where E is given by (4.10) and (4.11). This sum is nearly equal to the integral

$$\omega_0 \mu_s \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \exp\left\{-\frac{m_s c^2}{kT}\left\{\left[1+\frac{h^2}{4m_s^2c^2}\left(\frac{s'^2}{a^2}+\frac{t^2}{b^2}+\frac{u^2}{c^2}\right)\right]^{1/2}-1\right\}\right\} ds'dtdu,$$

and if we make the substitution $\alpha = hs'/2a$, $\beta = ht/2b$, $\gamma = hu/2c$ we have

$$Z_s = \omega_0 \mu_s \frac{8V}{h^3}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \exp\left\{-\frac{m_s c^2}{kT}\left\{\left[1+\frac{1}{m_s^2c^2}(\alpha^2+\beta^2+\gamma^2)\right]^{1/2}-1\right\}\right\} d\alpha d\beta d\gamma. \quad (4.12)$$

If we make a transformation to spherical polar co-ordinates with p as the radius vector, we obtain after two integrations

$$Z_s = \frac{8\times 4\pi V}{8h^3}\omega_0\mu_s \int_0^{\infty} \exp\left\{-\frac{m_s c^2}{kT}\left[\left(1+\frac{p^2}{m_s^2c^2}\right)^{1/2}-1\right]\right\} p^2 dp, \quad (4.13)$$

where the factor 1/8 appears because α, β and γ, being positive, are confined to 1/8 of the spherical space. If $\frac{m_s c^2}{kT}$ is large, the only important contributions to this integral come from small values of p and we obtain the ordinary result (4.0) for the classical case. But if $\frac{m_s c^2}{kT}$ is small, then the only important contributions to the integral come from large values of p for which, approximately, the exponent becomes $-pc/kT$. If we neglect trivial terms, we thus find for the case $kT >> m_s c^2$ the result

$$Z_s = \omega_0 \mu_s \frac{8\pi k^3 T^3 V}{h^3 c^3}. \quad (4.14)$$

If excited states of the nucleus can exist, we have instead of (4.14) the equation

$$Z_s = \mu_s \frac{8\pi k^3 T^3 V}{h^3 c^3} \sum {}_n \omega_n e^{-\epsilon_n/kT} \quad (4.15)$$

very nearly.

For the electron, if classical statistics can be used, case 1 (*a*) exists when $T << 6\times 10^9$, and case 1 (*b*) when $T >> 6\times 10^9$. For the proton, case 1 (*a*) exists when $T << 1{\cdot}1\times 10^{13}$, and case 1 (*b*) when $T >> 1{\cdot}1\times 10^{13}$. We shall rarely, in all probability, have to use the relativistic classical statistics for protons or for heavier particles; the electron is the only particle which we should expect to be likely to require the relativistic treatment.*

* For a general treatment of the relativistic quantum theory of perfect gases, see Jüttner, *Zeits. für Physik*, **47**, 542, 1928.

Case 2 (a)

(μ_s not small compared with 1, $kT \log \mu_s << m_s c^2$)

When $\mu_s < 1$ we have equation (F 1513) in the absence of excited levels :

$$Z_s = \omega_0 \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum_{j=1}^{\infty} \frac{\alpha(j)}{j^{3/2}} \mu_s{}^j, \tag{4.20}$$

where $\alpha(j) = (-1)^{j-1}/j$ for the electron, proton, neutron and so far as is known at present all nuclei of odd mass number, generally. For particles of even mass number $\alpha(j) = 1/j$. When excited nuclear levels exist we have equation (F 1514),

$$Z_s = \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum_{j=1}^{\infty} \frac{\alpha(j)}{j^{3/2}} \mu_s{}^j \sum_n \omega_n e^{-j\epsilon_n/kT}. \tag{4.21}$$

It appears on inspection that the series in the two equations above diverge when $\mu_s > 1$, although (4.20) certainly converges for $\mu_s = 1$. It appears on closer examination that the above equations break down for even particles when $\mu_s = 1$. By the definition of Z_s, Z_s for even particles contains the term $\omega_0 \log \frac{1}{1-\mu_s}$ corresponding to the lowest possible energy level, which is taken to be zero by definition ; and this term $\to \infty$ as $\mu_s \to 1$. Further, if we operate on this term with $\mu_s \frac{\partial}{\partial \mu_s}$ in accordance with equation (2.051) to obtain the number ${}_s\bar{a}_0$ of particles s in their lowest state, we find

$${}_s\bar{a}_0 = \frac{\omega_0 \mu_s}{1-\mu_s},$$

and hence, as $\mu_s \to 1$, ${}_s\bar{a}_0 \to \infty$. Accordingly, *for an even particle, μ_s can never be greater than unity*, and *when it is unity the density of this element* s *must be infinite in the assembly*. If we calculate the total number of particles of sort s, $\bar{X}_s$, by (4.20) and (2.051) we obtain the result for even particles

$$\bar{X}_s = \omega_0 \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum_{j=1}^{\infty} j^{-3/2} = \omega_0 \frac{(2\pi m_s kT)^{3/2}}{h^3} Vx$$

for $\mu_s = 1$, and an infinite result for $\mu_s > 1$. This shows that for even particles equations (4.20) and (4.21) fail when $\mu_s = 1$; and they should be used with caution when μ_s is very nearly equal to unity. We shall be fairly safe, however, in using (4.20) and (4.21) for even particles so long as $\bar{X}_s$ is less than or equal to

$$\omega_0 \frac{(2\pi m_s kT)^{3/2}}{h^3} Vx,$$

and in taking μ_s to be accurately unity when $\bar{X}_s$ exceeds the above quantity. The number * x in this paragraph is the sum of the series $\sum_1^{\infty} j^{-3/2}$, and is approximately equal to 2·60.

* x is the Riemann Zeta-function of real argument 3/2 : $\zeta(\frac{3}{2})$.

We need only consider the case $\mu_s > 1$ for odd particles, for the reasons given above. If there are no excited nuclear levels we have the asymptotic expansion (F 1546),

$$Z_s \sim \omega_0 \frac{(2\pi m_s kT)^{3/2}}{h^3} V \frac{2}{\sqrt{\pi}} \left\{ \tfrac{4}{15} (\log \mu_s)^{5/2} + \frac{\pi^2}{6} (\log \mu_s)^{1/2} + \ldots \right\}, \quad (4.22)$$

and if there are excited nuclear levels we have, approximately,

$$\left.\begin{aligned} Z_s \sim & \frac{(2\pi m_s kT)^{3/2}}{h^3} V \frac{2}{\sqrt{\pi}} \sum_{n=0}^{n'} \omega_n \left[\tfrac{4}{15} (\log \mu_s e^{-\epsilon_n/kT})^{5/2} \right. \\ & \left. + \frac{\pi^2}{6} (\log \mu_s e^{-\epsilon_n/kT})^{1/2} + \ldots \right] \\ & + \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^{5/2}} \mu_s{}^j \sum_{n'+1}^{\infty} \omega_n e^{-j\epsilon_n/kT}, \end{aligned}\right\} \quad (4.23)$$

where $\epsilon_{n'+1}$ is the energy of the lowest excited nuclear level for which

$$\mu_s e^{-\epsilon/kT} < 1.$$

For completeness we shall note here the accurate formula for Z_s which holds for all values of μ_s and for both odd and even particles,* so long as the relativistic change of mass is trivial for the important energies in the integral. It is (F 1536) in the absence of excited nuclear levels

$$Z_s = \omega_0 \frac{2\pi (2m_s)^{3/2} V}{h^3} \int_0^{\infty} \log f(\mu_s e^{-\epsilon/kT}) \epsilon^{1/2} d\epsilon \; ; \quad (4.24)$$

and if excited nuclear levels exist it is

$$Z_s = \omega_0 \frac{2\pi (2m_s)^{3/2} V}{h^3} \sum\nolimits_n \omega_n \int_0^{\infty} \log f[(\mu_s e^{-\epsilon_n/kT}) e^{-\epsilon/kT}] \epsilon^{1/2} d\epsilon, \quad (4.25)$$

where ω_n is the statistical weight of the nth excited nuclear level, ϵ_n is the energy of the level, and ϵ is the kinetic energy of the particle.

Case 2 (b)

(μ_s not small compared with unity, $kT \log \mu_s >> m_s c^2$; for electrons only, case probably does not occur for other particles)

Using equation (4.11) and the same transformations by which (4.13) was reached, we find

$$\begin{aligned} Z_s &= \sum \log (1 + \mu_s e^{-\epsilon/kT}) \\ &= \omega_0 \frac{4\pi V}{h^3} \int_0^{\infty} \log \left\{ 1 + \mu_s e^{-\frac{m_s c^2}{kT} \left[\left(1 + \frac{p^2}{m_s{}^2 c^2} \right)^{1/2} - 1 \right]} \right\} p^2 dp \end{aligned} \quad (4.30)$$

* With the exception of $\mu_s = 1$ for even particles, the reason for the failure in this case being that then the important values of ϵ are nearly equal to zero, in which case $\epsilon^{1/2} d\epsilon$ is not proportional (as it is usually) to the number of different sets of values s', t, u for the range of energies ϵ to $\epsilon + d\epsilon$.

for the case of an odd particle without excited nuclear levels. The important cases appear to be

$$kT \log \mu_s << m_s c^2 \quad \text{and} \quad kT \log \mu_s >> m_s c^2.$$

If $kT \log \mu_s << m_s c^2$, then the important contributions to the integral * come from small values of p, and the formulæ are obtained for the ordinary degenerate case. But when $kT \log \mu_s >> m_s c^2$, then the exponent is approximately equal to $-pc/kT$ for values of p which yield important contributions to the integral, and we have, nearly,

$$Z_s = \omega_0 \frac{4\pi V}{c^3 h^3} \int_0^\infty \log (1 + \mu e^{-\epsilon/kT}) \epsilon^2 d\epsilon, \tag{4.31}$$

where ϵ is the kinetic energy. If we set $\epsilon/kT = \epsilon'$ and then integrate by parts, we have

$$Z_s = \omega_0 \frac{4\pi V k^3 T^3}{3c^3 h^3} \int_0^\infty \frac{\epsilon'^3 d\epsilon'}{1 + e^{\epsilon'}/\mu_s},$$

and by Sommerfeld's method (F 1544) and (F 1545) we have the asymptotic expansion

$$Z_s \sim \omega_0 \frac{\pi V k^3 T^3}{3c^3 h^3} \{(\log \mu_s)^4 + 2\pi^2 (\log \mu_s)^2 + \ . \ . \ .\} \tag{4.32}$$

for an odd particle when $kT \log \mu_s >> m_s c^2$. If now we apply to Z_s the operator $\mu_s \frac{\partial}{\partial \mu_s}$ to find $\bar{X}_s$, we have

$$\bar{X}_s \sim \omega_0 \frac{4\pi V (kT \log \mu_s)^3}{3c^3 h^3},$$

and hence, replacing $kT \log \mu_s$ by $m_s c^2$, we find for protons or heavier particles that their density ρ in case 2 (b) exceeds $2{\cdot}9 \times 10^{15}$ grm./cm.^{3}. Unless we consider assemblies with such densities, we need never consider the relativistically degenerate case for odd particles other than electrons, and hence we need not work out here the formulæ in case 2 (b) for odd particles with excited nuclear levels.

For even particles, we have the integral, if we neglect excited nuclear levels,

$$Z_s = -\omega_0 \frac{4\pi V}{h^3} \int_0^\infty \log \left\{ 1 - \mu_s e^{-\frac{m_s c^2}{kT} \left[\left(1 + \frac{p^2}{m_s^2 c^2} \right)^{1/2} - 1 \right]} \right\} p^2 dp,$$

and here, since $\mu_s < 1$, the only way for the significant contributions to the integral to come from large values of p so as to yield a relativistic case, is for kT to be large compared with $m_s c^2$. But $\bar{X}_s$ is always an increasing

* There is of course the possibility that while $kT \log \mu_s << m_s c^2$, $kT >> m_s c^2$, and thus important contributions to the integral might come from large values of p. But this can only happen for *degenerate* statistics when μ_s is nearly equal to unity ; and we here ignore this trivial possibility ; except to notice that for this case the equilibrium properties may be worked out from the equation (4.30) itself.

function of μ_s, and for the case of $kT >> m_s c^2$, for μ_s of the order of 1/10 say, we have from equation (4.14) of the relativistic classical case,

$$\bar{X}_s = \omega_0 \mu_s \frac{8\pi k^3 T^3 V}{h^3 c^3}.$$

Replacing kT by $m_s c^2$, one finds that the density of even the lightest even element, $H_2{}^1$, would be greater than $2{\cdot}8 \times 10^{16}$ grm./cm.^{3}. Since the effect of levels of nuclear excitation is to increase the density beyond even this value, one probably never needs to consider the case 2 (*b*) for even particles at all.

§ 5. *Detailed Investigation of the Equilibrium Properties of the Assembly*

From the equations of §§ 2 and 4, all the important equilibrium properties of the assembly can be worked out. Since in stellar matter there is almost exact equality of positive and negative charge per unit volume, we have for the mean number of free * electrons per unit volume

$$\frac{\bar{X}_{el}}{V} = \frac{\rho}{\bar{U}} N,$$

where ρ is the density of matter in the assembly, and $\bar{U}$ is the mean molecular weight defined by

$$\bar{U} = \Sigma \bar{X}_s U_s / \Sigma \bar{X}_s N_s.$$

Here N is the number of atoms in a gram atom, and U_s is the atomic weight of atoms of sort s. Numerically,

$$\frac{\bar{X}_{el}}{V} = 6{\cdot}07 \times 10^{23} \frac{\rho}{\bar{U}}. \tag{5.00}$$

We now apply the equations (2.051) through (2.19) to the Z_s's which we have calculated in § 4, remembering to replace ξ by ξ' in the equations relating to electrons.

Ordinary Classical Case.—By (4.0) we have for the mean number of free electrons

$$\bar{X}_{el} = 2\xi' \frac{(2\pi m_{el} kT)^{3/2}}{h^3} V, \tag{5.01}$$

for the mean number of free protons

$$\bar{X}_P = 2\eta \frac{(2\pi m_P kT)^{3/2}}{h^3} V, \tag{5.02}$$

and for the mean number of particles of any sort † s by (4.01)

$$\bar{X}_s = \mu_s \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum\nolimits_n \omega_n e^{-\epsilon_n/kT}. \tag{5.03}$$

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* "Free" electrons are electrons which are not inside nuclei or neutrons.

† It should be remembered that for electrons, protons and probably for neutrons the spin is $\frac{1}{2}$, and thus $\omega_0 = 2$, with probably no excited levels.

The mean total energy of electrons is

$$\bar{E}_{el} = \tfrac{3}{2}\bar{X}_{el}kT, \tag{5.04}$$

of protons is

$$\bar{E}_P = \tfrac{3}{2}\bar{X}_P kT, \tag{5.05}$$

and of all the particles * s in the assembly, for which this case holds,

$$\bar{E}_s = \bar{X}_s Q_s + \tfrac{3}{2}\bar{X}_s kT + \bar{X}_s kT^2 \frac{\partial}{\partial T} \log \sum\nolimits_n \omega_n e^{-\epsilon_n/kT}. \tag{5.06}$$

In (5.06) the first term is energy of formation,† the second term is kinetic energy, and the third term is energy of nuclear excitation. The contributions to the pressure are

$$\left.\begin{aligned} p_{el} &= \bar{X}_{el}kT/V, \\ p_P &= \bar{X}_P kT/V, \\ p_s &= \bar{X}_s kT/V, \end{aligned}\right\} \tag{5.07}$$

and the number of particles in the nth state of nuclear excitation, with kinetic energies between ϵ and $\epsilon + d\epsilon$ is

$$n(\epsilon)d\epsilon = \mu_s \frac{2\pi(2m_s)^{3/2}}{h^3} V\omega_n e^{-(\epsilon_n+\epsilon)/kT}\epsilon^{1/2}d\epsilon. \tag{5.08}$$

This case exists accurately for any particle for which $\mu_s << 1$, $kT << m_s c^2$. It holds in particular for electrons when

$$2{\cdot}47 \times 10^5 \left(\frac{\rho}{\bar{U}}\right)^{2/3} << T << 6 \times 10^9.$$

Relativistic Classical Case.—For the reasons given in § 4, we probably do not need to study this case except for electrons. From equation (4.14) we have

$$\bar{X}_{el} = \xi' \frac{16\pi k^3 T^3}{c^3 h^3} V, \tag{5.09}$$

$$\bar{E}_{el} = 3\bar{X}_{el}kT \tag{5.10}$$

and

$$p_{el} = \bar{X}_{el}kT/V. \tag{5.11}$$

The number of electrons with energies between ϵ and $\epsilon + d\epsilon$ is for large values of ϵ

$$n(\epsilon)d\epsilon = \xi' \frac{8\pi V}{c^3 h^3} e^{-\epsilon/kT}\epsilon^2 d\epsilon. \tag{5.12}$$

For this case to hold accurately for electrons we must have $kT >> m_{el}c^2$, $\xi' << 1$. Numerically, these conditions are

$$T >> 6 \times 10^9 \quad \text{and} \quad T >> 3{\cdot}28 \times 10^7 \left(\frac{\rho}{\bar{U}}\right)^{1/3}.$$

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* See footnote in § 2 explaining the validity of the use of Q_s instead of Q_s' for case (*b*).

† In case (*b*) it includes an additive constant, for in our treatment the neutron, wherever it occurs, is given a lowest energy equal to $(c^2 - \varphi)(m_n - m_{el} - m_P)$.

Ordinary Degenerate Case. Odd Particles, μ_s Large.—By (4.22) we have for electrons

$$\bar{X}_{el} \sim \frac{4}{\sqrt{\pi}} \frac{(2\pi m_{el} kT)^{3/2}}{h^3} V \left\{ \tfrac{2}{3} (\log \xi')^{3/2} + \frac{\pi^2}{12} (\log \xi')^{-\frac{1}{2}} + \ldots \right\}, \quad (5.13)$$

$$\log \xi' \sim \frac{h^2}{8 m_{el}} \left(\frac{3 \bar{X}_{el}}{\pi V} \right)^{2/3} \frac{1}{kT} \left\{ 1 - \frac{\pi^2}{12} \left(\frac{8 m_{el} kT}{h^2} \right)^2 \left(\frac{3 \bar{X}_{el}}{\pi V} \right)^{-\frac{4}{3}} \right\}, \quad (5.131)$$

and for protons

$$\bar{X}_P \sim \frac{4}{\sqrt{\pi}} \frac{(2\pi m_P kT)^{3/2}}{h^3} V \left\{ \tfrac{2}{3} (\log \eta)^{3/2} + \frac{\pi^2}{12} (\log \eta)^{-\frac{1}{2}} + \ldots \right\}; \quad (5.14)$$

and for odd nuclei generally, including neutrons,

$$\left. \begin{aligned} X_s \sim \frac{2}{\sqrt{\pi}} \frac{(2\pi m_s kT)^{3/2} V}{h^3} \sum_{n=0}^{n'} \omega_n \Big[& \tfrac{2}{3} (\log \mu_s e^{-\epsilon_n / kT})^{3/2} \\ & + \frac{\pi^2}{12} (\log \mu_s e^{-\epsilon_n / kT})^{-1/2} + \ldots \Big] \\ & + \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum_{j=1}^{\infty} \frac{(-1)^{j-1} \mu_s{}^j}{j^{3/2}} \sum_{n'+1}^{\infty} \omega_n e^{-j\epsilon_n / kT} \end{aligned} \right\} \quad (5.15)$$

by (4.23). The mean total energies are

$$\bar{E}_{el} \sim \frac{6}{\sqrt{\pi}} \frac{(2\pi m_{el} kT)^{3/2}}{h^3} V kT \left\{ \tfrac{4}{15} (\log \xi')^{5/2} + \frac{\pi^2}{6} (\log \xi')^{1/2} + \ldots \right\} \quad (5.16)$$

$$\sim V \left\{ \frac{\pi}{40} \frac{h^2}{m_{el}} \left(\frac{3 \bar{X}_{el}}{\pi V} \right)^{5/3} + \frac{2\pi^3 m}{3h^2} \left(\frac{3 \bar{X}_{el}}{\pi V} \right)^{1/3} k^2 T^2 + \ldots \right\}, \quad (5.17)$$

and we do not bother to write down the corresponding expressions for the kinetic energies of other odd particles than electrons, further than to note that ${}_s\bar{E}_{\text{kin}} = \frac{3}{2} kT Z_s$, where Z_s is given by (4.23). We shall, however, write down the energies of formation and of nuclear excitation ; these are

$$\bar{E}_{\text{formation}} = \bar{X}_s Q_s \quad (5.171)$$

and

$$\bar{E}_{\text{nuclear}} \sim \frac{2}{\sqrt{\pi}} \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum_{n=0}^{n'} \omega_n \epsilon_n \left\{ \tfrac{2}{3} (\log \mu_s e^{-\frac{\epsilon_n}{kT}})^{\frac{3}{2}} + \frac{\pi^2}{12} (\log \mu_s e^{-\frac{\epsilon_n}{kT}})^{-\frac{1}{2}} + \ldots \right\}$$

$$+ \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^{3/2}} \mu_s{}^j \sum_{n'+1}^{\infty} \omega_n \epsilon_n e^{-j\epsilon_n / kT} \quad (5.175)$$

respectively.

The contributions of the odd particles other than electrons to the total pressure are trivial * compared with the pressure of the electrons ; for any sort of odd particle its contribution to the pressure is 2/3 times its mean total

* The mean total kinetic energies are inversely proportional to the masses, and directly proportional to the 5/3 power of the concentrations.

kinetic energy per unit volume, a result which can be obtained from Z_s. But the electron pressure is important; it is

$$p_{el} = \tfrac{2}{3}\bar{E}_{el}/V, \tag{5.18}$$

and numerically,* in terms of free electrons,

$$p_{el} \sim 2{\cdot}296 \times 10^{-27}\left(\frac{\bar{X}_{el}}{V}\right)^{5/3} + 5{\cdot}379 \times 10^{-6}T^2\left(\frac{\bar{X}_{el}}{V}\right)^{1/3}. \tag{5.181}$$

The number of electrons with kinetic energies between ϵ and $\epsilon + d\epsilon$ is

$$n(\epsilon)d\epsilon = \frac{4\pi(2m_{el})^{3/2}V}{h^3}\frac{\epsilon^{1/2}d\epsilon}{\dfrac{e^{\epsilon/kT}}{\xi'} + 1}, \tag{5.19}$$

and for all odd particles generally, the number in the nth excited nuclear level with kinetic energies between ϵ and $\epsilon + d\epsilon$ is

$$n(\epsilon)d\epsilon = \omega_n\frac{2\pi(2m_s)^{3/2}V}{h^3}\frac{\epsilon^{1/2}d\epsilon}{\dfrac{e^{(\epsilon_n+\epsilon)/kT}}{\mu_s} + 1}. \tag{5.20}$$

The conditions under which the above analysis holds accurately are that μ_s should be large compared with unity, and that $kT \log \mu_s$ should be small compared with $m_s c^2$. For electrons, these conditions are

$$T << 2{\cdot}98 \times 10^5(\rho/\bar{U})^{2/3}\,;\quad \rho/\bar{U} << 2{\cdot}80 \times 10^6.$$

Ordinary Degenerate Case (continued). Odd Particles, $\mu_s \leqslant 1$.—In this transition case between the classical case and the extreme degenerate case, for odd particles, we have from (4.21) for all odd particles

$$\bar{X}_s = \frac{(2\pi m_s kT)^{3/2}}{h^3}V\sum_{j=1}^{\infty}(-1)^{j-1}\frac{{\mu_s}^j}{j^{3/2}}\sum{}_n\omega_n e^{-j\epsilon_n/kT} \tag{5.21}$$

and

$$\bar{E}_s = \bar{X}_s Q_s + \tfrac{3}{2}\frac{(2\pi m_s kT)^{3/2}}{h^3}VkT\sum_{j=1}^{\infty}(-1)^{j-1}\frac{{\mu_s}^j}{j^{5/2}}\sum{}_n\omega_n e^{-j\epsilon_n/kT}$$
$$+\frac{(2\pi m_s kT)^{3/2}}{h^3}V\sum_{j=1}^{\infty}\frac{(-1)^{j-1}{\mu_s}^j}{j^{3/2}}\sum{}_n\omega_n\epsilon_n e^{-j\epsilon_n/kT}. \tag{5.22}$$

In the right-hand member of (5.22), the first term represents energy of formation,† the second term kinetic energy, $\bar{E}_{\text{kin}}$, and the third term energy of nuclear excitation. We have for the contribution to the pressure

$$p_s = \tfrac{2}{3}\bar{E}_{\text{kin}}/V, \tag{5.23}$$

and the energy distribution is given by (5.20), which holds for both the classical and degenerate case. This case exists when $\mu_s \leqslant 1$, and includes

* Calculated with $m_{el} = 9{\cdot}038 \times 10^{-28}$, $k = 1{\cdot}370 \times 10^{-16}$, $h = 6{\cdot}542 \times 10^{-27}$.

† With the usual conventions.

the classical case in the limit of small μ_s. It holds accurately for electrons when

$$6 \times 10^9 >> T \geqslant 2{\cdot}98 \times 10^5 \left(\frac{\rho}{U}\right)^{2/3}.$$

Ordinary Degenerate Case (continued). Even Particles.—Here $\mu_s \leqslant 1$. When $\mu_s = 1$ we have, by a previous discussion,

$$\bar{X}_s = \infty$$

and

$$\bar{E}_s = \pm\, \infty$$

because it contains the term $\bar{X}_s Q_s$. We have also from (4.25) that for the lowest nuclear level

$$n(\epsilon)d\epsilon = \omega_0 \frac{2\pi(2m_s)^{3/2}}{h^3} V \frac{\epsilon^{1/2} d\epsilon}{e^{\epsilon/kT} - 1},$$

except that for $\epsilon = 0$ this equation fails because the right-hand member does not become infinite with sufficient rapidity as* $\epsilon \to 0$. Pending a more careful study of the Einstein-Bose statistics for $\mu \to 1$, a reasonable procedure is to take μ_s to be unity whenever we know that

$$\bar{X}_s > \omega_0 \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum j^{-\frac{3}{2}}.$$

Since no μ_s for any other even nucleus can then be so much† as unity, we can calculate all the other $\bar{X}_s$'s without difficulty ; and obtain the density of s, and hence $\bar{X}_s$, by subtracting from the total density the densities of all the substances other than s.

When $\mu_s < 1$, then if $\bar{X}_s \leqslant \omega_0 \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum j^{-3/2}$ we can proceed without difficulty. By (4.21)

$$\bar{X}_s = \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum_{j=1}^{\infty} \frac{{\mu_s}^j}{j^{3/2}} \sum_n \omega_n e^{-j\epsilon_n/kT}, \tag{5.24}$$

$$\bar{E}_s = \bar{X}_s Q_s + \tfrac{3}{2} \frac{(2\pi m_s kT)^{3/2}}{h^3} kTV \sum_{j=1}^{\infty} \frac{{\mu_s}^j}{j^{5/2}} \sum_n \omega_n e^{-j\epsilon_n/kT}$$

$$+ \frac{(2\pi m_s kT)^{3/2}}{h^3} V \sum_{j=1}^{\infty} \frac{{\mu_s}^j}{j^{3/2}} \sum_n \omega_n \epsilon_n e^{-j\epsilon_n/kT}. \tag{5.25}$$

In (5.25) the first term represents energy of formation, the second term is $\bar{E}_{\text{kin}}$, the third is the energy of nuclear excitation. We have also

$$p_s = \tfrac{2}{3} \bar{E}_{\text{kin}}/V \tag{5.26}$$

* The case $\mu \to 1$ is interesting from a mathematical point of view, and the answer to such questions as how the molecular concentration, the pressure and the kinetic energy behave might repay further investigation.

† The reason for this was mentioned in § 3.

and for the nth excited nuclear level

$$n(\epsilon)d\epsilon = \omega_n \frac{2\pi(2m_s)^{3/2}V}{h^3} \frac{\epsilon^{1/2}d\epsilon}{\frac{e^{(\epsilon_n+\epsilon)/kT}}{\mu_s} - 1}. \tag{5.27}$$

Relativistically Degenerate Case. Electrons only.—From (4.32) we have

$$\bar{X}_{el} \sim \frac{8\pi k^3 T^3}{3c^3h^3} V\{(\log \xi')^3 + \pi^2 \log \xi' + \ldots\} \tag{5.28}$$

and

$$\log \xi' \sim \frac{ch}{2kT}\left(\frac{3\bar{X}_{el}}{\pi V}\right)^{1/3} - \frac{2\pi^2}{3}\frac{kT}{ch}\left(\frac{3\bar{X}_{el}}{\pi V}\right)^{-1/3} + \ldots \tag{5.29}$$

We have also

$$\bar{E}_{el} \sim \frac{2\pi k^4 T^4}{c^3h^3} V\{(\log \xi')^4 + 2\pi^2(\log \xi')^2 + \ldots\} \tag{5.30}$$

and

$$\bar{E}_{el} \sim V\left\{\frac{\pi ch}{8}\left(\frac{3\bar{X}_{el}}{\pi V}\right)^{4/3} + \frac{\pi^3}{3}\frac{k^2T^2}{ch}\left(\frac{3\bar{X}_{el}}{\pi V}\right)^{2/3} + \ldots\right\} \tag{5.301}$$

as well as

$$p_{el} \sim \bar{E}_{el}/3V. \tag{5.31}$$

Numerically,*

$$p_{el} \sim 2{\cdot}414 \times 10^{-17}\left(\frac{\bar{X}_{el}}{V}\right)^{4/3} + 3{\cdot}197 \times 10^{-16}T^2\left(\frac{\bar{X}_{el}}{V}\right)^{2/3}. \tag{5.311}$$

For the energy distribution,

$$n(\epsilon)d\epsilon = \frac{8\pi V}{c^3h^3} \frac{\epsilon^2 d\epsilon}{\frac{e^{\epsilon/kT}}{\xi'} + 1}, \tag{5.32}$$

from (4.31). This equation (5.32) holds for all values of ξ', so long as the relativistic change of mass is important, and includes equation (5.12) as the limiting form for small ξ'.

This case holds accurately when $\xi' >> 1$ and $kT \log \xi' >> m_{el}c^2$. It holds for electrons when $T << 5{\cdot}96 \times 10^7(\rho/\bar{U})^{1/3}$ and $\rho/\bar{U} >> 9{\cdot}89 \times 10^5$.

§ 6. *Conclusions*

With the equations of the previous sections, the constitution of stellar matter can be calculated (at least ideally) under given conditions of temperature and density. From a knowledge of the density and temperature, one may take an approximate value of the mean molecular weight $\bar{U}$ and by equation (5.00) obtain a value for the number of "free" electrons $\bar{X}_{el}$. Case (*a*) is the simplest to handle, and we shall consider it here. From a knowledge of $\bar{X}_{el}$ one can, knowing T, find $\xi = \xi'$; one can then (at least if one knows the packing fractions, the levels of nuclear excitation when they are important, and the appropriate statistical weights) calculate the $\bar{X}_s$'s for all possible particles for several choices of η until by trial and error or by interpolation the value of η is found which makes $\sum_s m_s\bar{X}_s$ equal to

* Here $c = 2{\cdot}998 \times 10^{10}$; the other values are the same as in (5.181).

the density ρ. From this result one may correct, if necessary, one's first assumed value of $\overline{U}$ and obtain a better approximation to the $\bar{X}_s$'s.

The difficulties connected at present with the computation of the abundances according to hypothesis (*a*) are largely those due to a lack of experimental data relating to packing fractions, nuclear levels and spins. The calculation of the $\bar{X}_s$'s when these are known appears to be tedious, for there are over one hundred and fifty sorts of particles, but by no means * prohibitively so ; and it may prove to be possible to shorten the arithmetic considerably without introducing significant errors by the use of a smoothed packing-fraction curve, and perhaps by other mathematical devices. While the packing fractions of many substances are already known, and while those not known by experiment may perhaps be obtained fairly accurately by interpolations making use of the curve of packing fractions, the lack of sufficient experimental knowledge about the spins and nuclear levels is a difficulty at present which there seems no way, in general, to circumvent.

Nevertheless, the following table may perhaps be of interest in showing the variation of composition with temperature in a simple hypothetical case ; while the case is artificial and has no physical significance, the phenomena which it shows—namely, the practical constancy of the composition until a certain high temperature is reached, and the subsequent rapid variation of composition with increasing temperature—may well carry over into the computations which it will be possible to make when the necessary experimental data shall have become available. Consider an hypothetical assembly in which the density of matter is 10 grm./cm.^{3}, at zero gravitational potential, and in which there can exist nothing but radiation, electrons and nuclei of H_1, He_4, O_{16}, Fe_{56} and $Ra\,B_{214}$. Then without too much labour it is possible to work out the following abundances,† in grm./cm.^{3} :—

Nucleus	$T=100$	$T=10^6$	$T=2\cdot10^9$	$T=3\cdot10^9$	$T=4\cdot10^9$
H_1	0	0	$1\cdot3\times10^{-7}$	·39	10
He_4	0	0	·016	9·61	10^{-5}
O_{16}	0	0	4×10^{-19}	10^{-16}	0
Fe_{56}	10	10	9·984	10^{-28}	0
$Ra\,B_{214}$	0	0	0	0	0

It will be noticed that for each temperature for which calculations ‡ were made, the matter in the assembly was composed almost entirely of one single

* It may be that often not many of the $\bar{X}$'s will be significantly large simultaneously.

† The nuclear spins of *He* and o are zero, and that of *H* is $\frac{1}{2}$; those of *Fe* and *Ra B* are taken arbitrarily to be o. Excited nuclear levels are neglected. The packing fractions are taken to be 77·8, 5·4, o, −9 and 2 for *H*, *He*, o, *Fe* and *Ra B* respectively.

‡ The figures in the table are probably nearly correct, but are not accurate ; the table was calculated by the use of a slide-rule.

element. It seems possible that this phenomenon also will sometimes carry over into the correct calculations.

Computations in accordance with hypothesis (*b*), where there are four variable parameters, appear to be necessarily more difficult than calculations in accordance with hypothesis (*a*).

It is perhaps hardly necessary to mention time-scales here, beyond the obvious comment that in any case the energy which a star can obtain from transmutations amounts roughly to 8×10^{18} ergs for each gram of hydrogen still remaining. This value corresponds to any final composition in which there is but little of any element remaining with a packing fraction comparable to that of hydrogen (the only elements of this sort so far known are hydrogen and neutrons). If the Sun were composed mostly of hydrogen, and continued to radiate energy at the present rate, it could hold out, with this source of energy alone, for approximately 10^{11} years.

There is a certain general inference with an astrophysical bearing which we can draw from the general solution of the statistical problem for both cases (*a*) and (*b*),* and which we mention in closing. If transmutations and gravitational contraction are ever the chief source of energy in a star, and if the matter in which the transmutations occur is in equilibrium as regards transmutations, then we may expect the *energy* (not, as in current † theories, the *rate of generation* of energy) of the stellar matter to depend upon the gravitational potential, the temperature, the density, and perhaps upon certain other quantities, of the nature of thermodynamic partial potentials, as well. The rate of liberation of energy will be governed by the rates ‡ at which the quantities are *changing*, as well as by the values of the quantities themselves.

Finally, I hope that the statistical theory here developed may prove to be of assistance to astrophysicists. It is a pleasure to thank my former

* Or indeed for any choice at all of the ultimate particles which an increase in our knowledge of nuclei may require us to make in the future.

† With the exception of Milne's hypothesis already referred to. Milne's hypothesis must in any case be generalized to include transmutations other than the formation of neutrons from electrons and protons, and his law $\epsilon = F(dT/dt)$ then becomes the special case of the law to which we are led, when we allow only the temperature to vary.

‡ If the important interior regions of the stars were becoming *hotter* instead of colder, it is hard to see how energy could be obtained from the transmutations at all—unless, indeed, the density were increasing so rapidly as to displace the equilibria in the sense of diminishing amounts of the light elements, *despite* the increasing temperature. Whether or not this could be possible will require further investigation to answer. The conditions of stability imposed by the equilibria of transmutations will require investigation in any case, before one will be able to answer definitely such questions as whether a star, which is obtaining energy from the displacement of the equilibria, must be becoming hotter or colder, larger or smaller. It looks very much as though a star in which the transmutations were *not* in equilibrium would be over-stable if it obtained much energy from the transmutations.

Whether or not stars obtain their energy to an appreciable extent from transmutations, the equilibrium theory should still tell us something about the abundances, and about the contributions (positive or negative !) made by the transmutations to the star's radiation, provided only that the matter is sufficiently well insulated.

teacher, Professor R. H. Fowler, for the suggestions * and encouragement which he has sent to me by letter, and to acknowledge the interest of Professor E. C. Kemble of Harvard University in my work. I wish to thank Miss C. H. Payne and Dr. D. H. Menzel of the Harvard College Observatory for the information about observational data with which they have supplied me in conversations. I also acknowledge gratefully the grant which I have received from the National Research Council, Washington, D.C., and without which this work could not have been carried out.

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Cambridge, Mass.:
1933 *March* 15.

A NOTE ON THE LIBERATION OF ENERGY BY TRANSMUTATIONS OF NUCLEI IN THE STARS.

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(Communicated by C. H. Payne)

In the preceding paper a statistical study was made of assemblies in equilibrium, in which all possible transmutations of the various nuclei occurred. It is easy to see how such an assembly will liberate energy as it is allowed to cool, at constant volume, at a rate so slow that the equilibria at any one time during the process of cooling are nearly perfectly established. The mean total energy of the assembly at a temperature T is

$$\bar{E} = \bar{E}_r + \sum_s \bar{E}_s \tag{1}$$

in the notation of the preceding paper; except that now we shall handle the electrons, neutrons and protons more briefly in the discussion by treating them as ordinary nuclei. For the electron M is 0 and N is -1; for the proton M is 1 and N is 1; for the neutron M is 1 and N is 0. Then the energy liberated per second by the assembly as it cools slowly at the rate $-dT/dt$, at constant volume, is

$$-\frac{d\bar{E}}{dt} = -\frac{d\bar{E}_r}{dT}\frac{dT}{dt} - \mathbf{C}\frac{dT}{dt}, \tag{2}$$

where

$$\frac{d\bar{E}_r}{dT} = \frac{32\pi^5 k^4}{15c^3h^3} VT^3,$$

and where $\mathbf{C}$ is the "specific heat" of the matter, given by

$$\mathbf{C} = \sum_s \frac{d\bar{E}_s}{dT}. \tag{3}$$

* Notably that I should consider hypothesis (*b*) and in it ignore the β radioactive elements.

Here $\sum_s \frac{dE_s}{dT}$ must be calculated subject to the conditions that for hypothesis (a) the total number of electrons D_{el} and of protons D_P must be maintained constant; while for hypothesis (b) the total number of neutrons D_n must be kept constant as well. There is of course no "generation" of energy by the assembly when its temperature and volume are maintained strictly constant. As it cools, the assembly behaves merely as a hot body with a considerable specific heat; it is for this reason better to speak of the "liberation" of energy by such an assembly than it is to speak of the "generation" of energy by it. We shall give the equations for hypothesis (a). The equations in case (b) are here omitted; they are equally straightforward but slightly more cumbersome. For a given assembly there is in case (a) only one independent variable which we take to be T; ξ and η are determined as functions of T by the equations giving the total numbers of electrons and protons:

$$\left.\begin{aligned} D_{el} &= \sum_s (M_s - N_s)\mu_s \frac{\partial Z_s}{\partial \mu_s} \\ D_P &= \sum_s M_s \mu_s \frac{\partial Z_s}{\partial \mu_s}. \end{aligned}\right\} \tag{4}$$

The parameter μ_s, being defined in terms of ξ, η and T, is a function of T alone; so ultimately are Z_s, $\bar{X}_s$ and $\bar{E}_s$. But for purposes of computation it is best to obtain equations involving the partial derivatives of these quantities with respect to μ_s and T alone, as though μ_s and T were two independent variables. With this convention we have

$$\begin{aligned} \frac{d}{dT} f(\mu, T) &= \frac{\partial f}{\partial T} + \frac{\partial f}{\partial \mu}\frac{d\mu}{dT} \\ &= \frac{\partial f}{\partial T} + \mu \frac{\partial f}{\partial \mu}\left(\frac{Q_s}{kT^2} + \frac{M_s - N_s}{\xi}\frac{d\xi}{dT} + \frac{M_s}{\eta}\frac{d\eta}{dT}\right), \end{aligned}$$

and so, since

$$\bar{E}_s = Q_s \mu_s \frac{\partial Z_s}{\partial \mu_s} + kT^2 \frac{\partial Z_s}{\partial T} \tag{4'}$$

we have

$$\begin{aligned} C = \sum_s \Bigg[& Q_s \mu_s \frac{\partial^2 Z_s}{\partial \mu_s \partial T} + k\frac{\partial}{\partial T}\left(T^2 \frac{\partial Z_s}{\partial T}\right) \\ & + \mu_s \left(Q_s \frac{\partial}{\partial \mu_s}\mu_s \frac{\partial Z_s}{\partial \mu_s} + kT^2 \frac{\partial^2 Z_s}{\partial \mu_s \partial T}\right)\left(\frac{Q_s}{kT^2} + \frac{M_s - N_s}{\xi}\frac{d\xi}{dT} + \frac{M_s}{\eta}\frac{d\eta}{dT}\right)\Bigg]. \end{aligned} \tag{5}$$

Here ξ and η are determined by the equations (4), and $\frac{d\xi}{dT}$ and $\frac{d\eta}{dT}$ are determined by the two equations

$$\left.\begin{aligned} 0 &= \sum_s [M_s - N_s]\left[\mu_s \frac{\partial^2 Z_s}{\partial \mu_s \partial T} + \mu_s \frac{\partial}{\partial \mu_s}\left(\mu_s \frac{\partial Z_s}{\partial \mu_s}\right)\left(\frac{Q_s}{kT^2} + \frac{M_s - N_s}{\xi}\frac{d\xi}{dT} + \frac{M_s}{\eta}\frac{d\eta}{dT}\right)\right] \\ 0 &= \sum_s M_s \left[\mu_s \frac{\partial^2 Z_s}{\partial \mu_s \partial T} + \mu_s \frac{\partial}{\partial \mu_s}\left(\mu_s \frac{\partial Z_s}{\partial \mu_s}\right)\left(\frac{Q_s}{kT^2} + \frac{M_s - N_s}{\xi}\frac{d\xi}{dT} + \frac{M_s}{\eta}\frac{d\eta}{dT}\right)\right]. \end{aligned}\right\} \tag{6}$$

It is also easy to predict how the assembly will liberate energy as it is slowly compressed at constant temperature. The rate of liberation of energy is obviously equal to

$$-\frac{d\bar{E}}{dt}=-\frac{d\bar{E}_r}{dV}\frac{dV}{dt}-\frac{d\Sigma_s\bar{E}_s}{dV}\frac{dV}{dt},$$

where

$$\frac{d\bar{E}_r}{dV}=\frac{8\pi^5k^4T^4}{15c^3h^3}$$

and

$$\frac{d}{dV}\Sigma_s\bar{E}_s=\Sigma\left\{Q_s\mu_s\frac{\partial^2Z_s}{\partial\mu_s\partial V}+kT^2\frac{\partial^2Z_s}{\partial T\partial V}+\mu_s\left(Q_s\frac{\partial}{\partial\mu_s}\mu_s\frac{\partial Z_s}{\partial\mu_s}+kT^2\frac{d^2Z_s}{\partial\mu_s\partial T}\right)\left(\frac{M_s-N_s}{\xi}\frac{d\xi}{dV}+\frac{M_s}{\eta}\frac{d\eta}{dV}\right)\right\}.$$

The quantities ξ and η are defined as before by the equations (4) in terms of T and V, while $\frac{d\xi}{dV}$ and $\frac{d\eta}{dV}$ are defined by the equations

$$\left.\begin{aligned}0&=\Sigma_s(M_s-N_s)\left\{\mu_s\frac{\partial^2Z_s}{\partial\mu_s\partial V}+\left(\mu_s\frac{\partial}{\partial\mu_s}\mu_s\frac{\partial Z_s}{\partial\mu_s}\right)\left(\frac{M_s-N_s}{\xi}\frac{d\xi}{dV}+\frac{M_s}{\eta}\frac{d\eta}{dV}\right)\right\},\\0&=\Sigma_sM_s\left\{\mu_s\frac{\partial^2Z_s}{\partial\mu_s\partial V}+\left(\mu_s\frac{\partial}{\partial\mu_s}\mu_s\frac{\partial Z_s}{\partial\mu_s}\right)\left(\frac{M_s-N_s}{\xi}\frac{d\xi}{dV}+\frac{M_s}{\eta}\frac{d\eta}{dV}\right)\right\}.\end{aligned}\right\}\quad(7)$$

The rate of liberation of energy when both T and V are changing slowly is, of course,

$$-\frac{d\bar{E}}{dt}=-\left[\frac{d\bar{E}_r}{dT}\right]_V\frac{dT}{dt}-C\frac{dT}{dt}-\left[\frac{d\bar{E}_r}{dV}\right]_T\frac{dV}{dt}-\left[\frac{d\Sigma_s\bar{E}_s}{dV}\right]_T\frac{dV}{dt}.$$

Finally, let the assembly, of volume V', be in equilibrium at a temperature T', and let any change whatever occur, so that after the assembly is once more in equilibrium the new temperature and volume will be T'' and V'' respectively. Then the energy lost by the assembly during the change is $\bar{E}'-\bar{E}''$, where $\bar{E}'$ and $\bar{E}''$ are obtained from equations (1), (4) and (4′) (setting $T=T'$, $V=V'$ in the calculation of $\bar{E}'$, and setting $T=T''$, $V=V''$ in the calculation of $\bar{E}''$).

For stellar matter, D_{el} should be taken equal to D_P.*

The Jefferson Physical Laboratory,
Cambridge, Mass.:
1933 *March* 30.

* In a *star* there will of course be changes of gravitational potential energy to be taken into account besides the changes in $\bar{E}_r+\Sigma_s\bar{E}_s$ considered in this note.

THE EQUILIBRIUM OF TRANSMUTATIONS IN STARS IN WHICH TRANSMUTATIONS ARE AN IMPORTANT SOURCE OF ENERGY.

T. E. Sterne, Ph.D.

(Communicated by R. H. Fowler, F.R.S.)

In papers which were recently communicated to the Society * the writer presented the mathematical theory of the abundances of nuclei in an assembly in equilibrium, in which all possible transmutations occurred. Equations were obtained not only for the abundances of the various nuclei, but also for the rate of liberation of sub-atomic energy by the transmutations as the temperature and density of the assembly slowly changed.† It was shown that even in a star such as V Puppis A, which radiates energy at the exceptionally large rate of approximately 1000 ergs per gram per second, matter at any place inside the star was so well insulated by the matter surrounding it that it lost probably far less than 1 per cent. of its energy in a time of the order of 500 years. Since the matter was so well insulated, it appeared that the equilibrium of transmutations might have had time to become established in the stars.

At first sight there would appear to be two alternative ways in which appreciable contributions could be made, by transmutations, to the energy radiated by a star. The important contributions made by transmutations to this energy must come in any case from the disappearance of hydrogen of mass 1 and 2, or neutrons, since these have packing fractions which are considerably greater ‡ than the packing fractions of other kinds of nuclei. One way in which these light elements could disappear would be for the temperature to be such that the capture of these nuclei in transmutations occurred slowly, and liberated energy at so slow a rate that it could be used up, to a considerable extent, by the star's radiation into space. It appears from the table on a subsequent page of this paper that the temperatures necessary for this to happen could well be of the general order of magnitude of many of the central temperatures in Eddington's models—about fifty million degrees. The equilibrium theory of transmutations would not be

* *M.N.*, 93, 736, 767, 1933.

† Sub-atomic energy may be liberated under certain circumstances by an increasing compression of the assembly. A falling temperature is not necessarily essential in order that the assembly may liberate energy.

‡ The packing fraction of hydrogen is 77·8, that of neutrons, from the experiments of Chadwick, is 67. The packing fraction of hydrogen of mass 2 has been determined by Bainbridge to be 67·5 (*Phys. Rev.*, 42, 1, 1932).

applicable to processes of this kind,* for the reverse processes—the manufacture of hydrogen and neutrons—would not take place at nearly the same speed as the direct processes, at these temperatures, until almost all of the hydrogen and neutrons had disappeared. It will appear that this manner of generating energy would be very likely to cause a star to be over-stable.

The other possible way in which transmutations could make appreciable contributions to the energy would be for the hydrogen or neutrons to be liberating energy, through being captured by other nuclei, more rapidly than this energy could be used up by radiation into space. The only way in which this could occur in a star in a steady state would be for the reverse processes to occur so frequently as to absorb the excess energy ; and unless they occurred indeed with nearly the same speeds as the direct processes, the serious objection which appears to hold against the first manner of generating energy, *i.e.* that it would lead to over-stability, would hold against this manner with equal force. Thus, to avoid the difficulties associated with the first manner of generating energy, the second way must involve nearly equal speeds of disappearance and manufacture for hydrogen and neutrons. But this necessitates that the equilibrium theory should be able to predict fairly accurately the relation between the temperature and the abundances of the hydrogen and neutrons. Using the equilibrium theory, it is found that for there to be appreciable amounts of hydrogen or neutrons the temperature must be of the order of 10^9 or more ; unless these appreciable abundances exist there is not much energy remaining to be liberated by transmutations. At these temperatures, further, the table shows that transmutations are taking place with so great a frequency that the equilibrium considerations must hold with considerable accuracy. As energy is contributed by the transmutations to the energy radiated by the star there will be a gradual displacement of the equilibria caused by the gradual changes of temperature and density, but the drift is of extreme slowness compared with the speeds of direct and reverse transmutation. We shall begin our discussion by investigating the speeds of disappearance of hydrogen and neutrons in an assembly which is not in equilibrium.

§ 1. *The Liberation of Energy by the Capture of Light Particles in a Non-Equilibrium State*

Of the three kinds of nuclei, neutrons, hydrogen of mass 1 and hydrogen of mass 2, experiments and theory alike suggest that the neutrons will tend to disappear the most rapidly. The theoretical reason for the rapidity of disappearance of neutrons is that they do not possess any net charges, and can thus penetrate other nuclei, particularly heavy ones, with greater ease than can charged particles. Next in rapidity of disappearance are hydrogen of mass 1 and hydrogen of mass 2, according to Gamow's theory †

* The processes suggested by Atkinson appear to have been of this non-equilibrium variety (*Astrophysical Journal*, **73**, 250, 1931 ; **73**, 308, 1931).

† *Zeit. für Physik*, **52**, 510, 1928.

of the penetration of nuclei by charged particles, the general correctness of which, for α particles and protons, has been verified * experimentally. Since we are interested merely in showing that the speed of disappearance of any of these particles, in the absence of reverse processes, increases in a most rapid fashion with increasing temperature, it does not much matter which one of these particles we select for calculating the speed of disappearance. If we prepare a table showing the rate of disappearance of hydrogen of mass 1, the general features of this table would be reproduced faithfully in other tables, if we constructed them, showing the rates of disappearance of the other sorts of particles in the absence of equilibrium. The temperatures for corresponding rates of disappearance would differ, but the sensitiveness of the speeds to small changes in temperature would remain. It would probably be unfair to consider the disappearance of neutrons by Gamow's theory, because that would predict a probability of unity that a neutron which hit a nucleus would penetrate it, and this predicted probability is perhaps too large. Further, we do not know of the existence of neutrons in appreciable numbers in the stars. We do know that there is hydrogen, however, and we may apply Gamow's theory to the disappearance of hydrogen of mass 1 with considerable confidence as to the general correctness, at least as to orders of magnitude, of the frequencies of capture so calculated,† and with even more confidence as to the general correctness of the *relative* frequencies so calculated for different temperatures, which are what we are really interested in.

Experiments ‡ indicate that protons can be captured by lithium nuclei of mass 7 more rapidly than by other nuclei. It really does not affect the rigour of our arguments which element we take in our calculations as the one which is most important in capturing the hydrogen. The hydrogen would disappear mainly by being captured by an element of a kind which was determined by the abundances of the various elements and by considerations of probability. If appreciable amounts of lithium of mass 7 were present, this element would be the most effective in removing the hydrogen from the assembly at such low temperatures that equilibrium was not established. If we carry out our calculations for lithium of mass 7 and find a rapid variation of the rate of capture with temperature, we may be

* For α particles, *vide* Gamow, *loc. cit.* For protons, *vide* Lawrence, Livingston and White, *Phys. Rev.*, **42**, 150, 1932 ; Henderson, *Phys. Rev.*, **43**, 98, 1933.

† The general correctness of the speeds, calculated by Gamow's theory, of the disappearance of hydrogen of mass 1 through capture by lithium of mass 7, has been verified experimentally by Henderson, *loc. cit.* The observed variation of the rate of capture with the energy of the protons is very closely equal to the variation expected according to Gamow's theory of penetration, but the rate of capture for a given energy appears to be from a tenth to a hundredth of the expected rate of penetration. Gamow's theory does not really give the probability of capture, but merely the probability that if a proton strikes a nucleus it will penetrate it. If these two probabilities are even approximately proportional to each other, the arguments in the text are unaffected. The observations of Henderson, and of Cockroft and Walton, show that these probabilities are in fact proportional to each other over the whole wide range of energies which was covered by their experiments.

‡ Cockroft and Walton, *Proc. Roy. Soc.*, A, **137**, 229, 1932.

confident from Gamow's theory that this rapid variation would still exist for processes of capture by a heavier variety of nucleus in case not enough lithium were present for the captures by lithium to be the most numerous ones.

According to Gamow's theory * the probability that a particle of charge Ze, mass m and effective energy E will penetrate a nucleus of charge $Z'e$, when it strikes it, is

$$W_1^* = \exp\left(-\frac{4\pi\sqrt{(2m)}}{h\sqrt{E}}ZZ'e^2\mathcal{J}_k\right)$$

where $\mathcal{J}_k$, which varies slowly with the energy of the collision, is defined by the integral

$$\mathcal{J}_k = \int\sqrt{\left(\frac{1}{\rho}-\frac{k^2}{\rho^3}-1\right)}d\rho$$

taken over the range of ρ for which the integrand is real. The number k is defined by the relation

$$k = \frac{r_m E}{\sqrt{3}ZZ'e^2}$$

where r_m is the "radius" of the struck nucleus, given sufficiently closely by

$$r_m = 1{\cdot}21 \times 10^{-13}A^{1/3},$$

where A is the atomic weight. Gamow's paper contains a graph, obtained by numerical integration, with which one can find $\mathcal{J}_k$ when k is known.

In our case the proper effective energy to use for a collision must take into account the motion of the lithium nucleus as well as the motion of the proton, and we take this energy to be the total kinetic energy of the colliding particles relative to their centre of mass. This energy, for a given temperature, will be different for different encounters, and for classical statistics the number † of encounters per unit volume per unit time for which the kinetic energy of the motion relative to the centre of mass lies between E and $E+dE$ is ‡

$$\frac{2\nu\nu'\sigma}{\pi(kT)^{3/2}}\left(\frac{2\pi(m+m')}{mm'}\right)^{1/2}e^{-E/kT}EdE$$

where ν and ν' are the molecular densities and σ is the effective cross-section for collisions. The integrations which are necessary have been performed by Atkinson and Houtermans,§ who obtained expressions for the number of penetrations per unit volume per unit time, neglecting the motion of the struck nucleus and setting $\mathcal{J}_k$ equal to its maximum value $\pi/2$.

* Gamow, *loc. cit.* See also Cockroft and Walton, *loc. cit.*

† Fowler, *Statistical Mechanics*, equation 1182.

‡ By using the classical statistics we shall be underestimating the energies of the collisions if anything, and thus underestimating the number of penetrations per second at a given temperature. This would leave our arguments unaffected.

§ *Zeit. für Physik*, 54, 656, 1929. The important contributions to the integrals come from values of E considerably greater than $2kT$, the mean energy of a collision.

Their result can be corrected easily to take into account the motion of the struck nucleus, and then becomes

$$N = \frac{2vv'\sigma}{\pi}\left(\frac{2\pi kT(m+m')}{mm'}\right)^{1/2} 2\xi\sqrt{\frac{\pi}{3}}(\xi^2+1)e^{-3\xi^2}$$

where

$$\xi = \left(\frac{\pi^2 ZZ'e^2\sqrt{(2m)}}{h\sqrt{(kT)}}\right)^{1/3}.$$

Here N is the number of penetrations * per unit volume per unit time. The results of the computations are given in the following table,† where ρ is the density of Li_7:—

T (°k)	Mean time for penetration t	($\epsilon \propto T^s$) s
10^6	$\frac{1{\cdot}1 \times 10^{23}}{\rho}$ years	29·4
5×10^6	$\frac{1{\cdot}6 \times 10^{7}}{\rho}$ years	17·2
10^7	$\frac{440}{\rho}$ years	13·7
4×10^7	$\frac{53}{\rho}$ minutes	8·6
10^8	$\frac{3{\cdot}4}{\rho}$ seconds	6·4
5×10^8	$\frac{1{\cdot}12 \times 10^{-3}}{\rho}$ seconds	3·8
10^9	$\frac{1{\cdot}07 \times 10^{-4}}{\rho}$ seconds	3·0
5×10^9	$<\frac{2{\cdot}52 \times 10^{-6}}{\rho}$ seconds	...

§ 2. *The Capture of Protons by Li_7 Nuclei in a Non-Equilibrium State*

In this table "the mean time for penetration to occur," t, is that time during which each proton would have a good chance of penetrating at least one lithium nucleus. If a process of this sort supplied energy for a star gradually and not in the form of an explosion, the mean time for pene-

* For very high temperatures this should give values of N which are in error by being too small, owing to the neglected slow decrease of $\mathcal{J}_k$ from its maximum value, with increasing T. This will leave unaffected the rigour of our arguments concerning speeds of disappearance and the quantity s. We shall be underestimating N, but for the important temperatures our calculated values will be sufficiently accurate.

† Following Cockroft and Walton, *loc. cit.*, we take σ to be 10^{-25} cm.^{2}.

tration would have to be of greater order probably than $8 \times 10^{18} fM/L$ seconds, since a gram of hydrogen can supply approximately 8×10^{18} ergs of energy when it undergoes transmutation. Here f is the proportion of hydrogen in a star, M is the mass of the star, and L is the luminosity in ergs per second. The time t must be of this order, because after this time has elapsed the hydrogen, as a source of energy, will have practically disappeared. Since L/M is seldom more than 1000, t should be taken to be greater perhaps than 10^8 years. We have here supposed that penetration is equivalent to capture; but the probability of capture when penetration occurs has been shown experimentally * to be a tenth or a hundredth at least, and to be accurately constant. If it is not unity, our scale of temperatures will be shifted somewhat, but our arguments will still hold. With any reasonable value for the density ρ of lithium the relevant part of the table for a star which obtained its energy in this fashion would have to be well towards the top in order for t to fall between the orders of magnitude of $1{\cdot}1 \times 10^{23}/\rho$ years and $10^{-4}/\rho$ years.

For this process the rate of generation of energy, ϵ, is proportional to the number of penetrations per second. For a small range of temperature we may think of the law of generation as being given in the form $\epsilon \propto T^s$. Hence $s = \frac{T}{\epsilon}\frac{d\epsilon}{dT} = \frac{T}{N}\frac{dN}{dT}$ and can be calculated from the equation for N. Thus the numbers in the last column of the table represent the exponents of T in the expression $\epsilon \propto T^s$, and we see that if the generation of energy by the capture of protons by lithium nuclei took place in the part of the table where it would have to if this process were the important one for generating energy, the rate of generation of energy would vary with the temperature proportionately to T^s where s lay between 8·6 and 29·4. Now Eddington † has shown that a star will be over-stable if the generation of energy varies with the temperature more rapidly than T^3. By saying that a star will be over-stable one means that a slight displacement from an equilibrium configuration will be followed by oscillations of increasing violence. Eddington obtained this criterion by consideration of a particular stellar model, and we see that that model would be violently over-stable with the process of energy generation which we have been considering. Our leeway is, moreover, so great that we feel strongly tempted to conclude that no star could avoid over-stability, whatever its distributions of density and temperature, if the rate of generation of energy were as sensitive to temperature changes as the rate which we have found.

The difficulties connected with the liberation of energy through the capture of protons by lithium nuclei in a non-equilibrium state cannot be avoided by supposing, for instance, that there is so little lithium present that this process cannot occur. If transmutations are to furnish energy by non-equilibrium captures, then there would in this case have to be some other element, such as beryllium for instance, which would be the most important element in capturing protons. A table prepared for this process would

* See the footnote on a previous page.

† *The Internal Constitution of the Stars*, Chapter VIII.

reproduce the general features of the table for lithium, and the process would lead to over-stability.* Nor can the difficulty be avoided by supposing that other particles than protons are being captured and constitute the chief source of energy. Whatever the important process is, a table prepared for it will reproduce the general features of the table for the capture of protons by lithium, although perhaps with a displaced temperature scale, so long as the process is not nearly in equilibrium. Thus the possibility that an important source of stellar energy could be the gradual disappearance of the elements of large packing fraction, through transmutations which are not accompanied by the reverse processes of their manufacture, appears to be ruled out.

§ 3. *The Reaction Speeds in Equilibrium States*

The rapid variation of the generation of energy with increasing temperature can only be avoided if the direct and reverse transmutations, which involve the disappearance and manufacture of the elements of large packing fraction, occur with comparable speeds. But in this case the abundances of the elements in question will be given fairly accurately by equilibrium considerations, and the presence of appreciable abundances of these elements demands that the temperature should be of the order of 10^9 °K or more.† If the abundance of lithium is even so small as 10^{-6} grams/cm.^{3}, we see from our table that the hydrogen will on the average be captured each 107 seconds by lithium nuclei, and it will be remade at the same rate. At a temperature of 5×10^9, for a density of lithium of 10^{-3} grams/cm.^{3}, the time would be less than 3×10^{-3} seconds. Neutrons will probably be causing transmutations more rapidly than protons, if anything; and the protons will really be causing transmutations more rapidly than we have stated, because they will cause transmutations in other elements than lithium. If the radiation of a star were obtained entirely from the energy of transmutations, and if the rate of radiation were even so large as 10,000 ergs per gram of hydrogen per second, it would still take more than 10^7 years for the hydrogen to be exhausted. Thus the reaction times are extremely short compared with the time required for the equilibria to be appreciably displaced by the changing state of the star; in fact the reaction times are so short that we should expect our equilibria at these high temperatures to follow fairly faithfully even comparatively rapid changes in the temperature and density of the stellar matter.

* In fact it would lead to increased over-stability. Suppose that the captures are caused by nuclei of atomic number $3a$ instead of by lithium. At a temperature Ta^2 the rate of capture would be a times as great as for lithium at the temperature T, and s would be the same as for lithium at the temperature T. To obtain the same rate of liberation of energy as for lithium the temperature in the case of the element of atomic number $3a$ would have to be somewhat lower than Ta^2, and thus s would be greater than in the case of lithium. Provided that ξ^2 is not too close to unity, s is approximately equal to ξ^2. The increase of σ with Z' strengthens this argument.

† See Sterne, *loc. cit.*, for the equations. We suppose that the density is not so great, for instance, as 10^{10} grams/cm.^{3}. Such enormous densities would require the temperature to be higher than 10^9 °K.

It is obvious that the difficulty of over-stability, which is so serious an objection to the notion that energy may be liberated by non-equilibrium processes of transmutation in stars, does not hold against the notion that energy may be obtained from transmutations which are in equilibrium.* One feels strongly tempted to conclude that if transmutations are an important source of stellar energy, then they must be in equilibrium.†

I acknowledge gratefully the grant which I have received from the National Research Council,‡ without which these investigations could not have been carried out.

The Jefferson Physical Laboratory,
Cambridge, Mass.:
1933 *June* 6.

CALCULATION OF EPHEMERIDES IN NEARLY PARABOLIC ORBITS.

Jens P. Möller.

(Communicated by E. Strömgren)

Using the notation generally employed we define the following auxiliary quantities :—

$$P=(t-T)q^{-\frac{3}{2}}, \qquad \epsilon=\frac{1-e}{2} \quad \text{and} \quad \sigma=\frac{\sin\frac{E}{2}}{\sqrt{\epsilon}}. \tag{1}$$

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* *Note added in proof, September* 19.—There remains the possibility of instability, which will require further investigation. A small adiabatic change may be represented by the equation $p \propto \rho^{\kappa}$, where the constant κ must be calculated by the equilibrium theory of the abundances, and is *not* a ratio of specific heats. The possible stable states are those for which $\kappa > 4/3$ in the important regions of the star ; when $\kappa < 4/3$ the configuration will be unstable. I have studied the matter enough to see that there are some stable states and some unstable states ; it is barely possible that sudden transitions from one stable state through an unstable state to another stable state might correspond to nova phenomena.

† I am indebted to Professor Russell for the following alternative argument to the one in the text above, to show that the transmutations must be in equilibrium if they supply energy to a star in a steady state. If there are greater abundances of the energy-producing elements (hydrogen, neutrons) in a star than the amounts which would exist in the equilibrium state at that temperature, then energy will be generated at a rate which increases so rapidly with increasing temperature as to make the star over-stable. This follows from the arguments in the text above. If, on the other hand, there are smaller abundances of the energy-producing elements than the equilibrium amounts, then the star will undergo an unusually rapid Kelvin contraction, because the transmutations will be absorbing energy. Thus the stability of a star, in a steady state, is consistent only with the equilibrium of the transmutations occurring within it.

‡ Washington, D.C.

Kepler's equation can then be written as follows :—

$$\frac{k}{\sqrt{2}} P = \sigma \cos\frac{E}{2} + \sigma^3 \frac{E - \sin E}{4 \sin^3 \frac{E}{2}} \tag{2}$$

and for the co-ordinates in the plane of the orbit we get the expressions

$$\bar{x} = q\,(1 - \sigma^2)\,;$$
$$\bar{y} = 2q\sqrt{1-\epsilon}\;\sigma\sqrt{1-\epsilon\sigma^2}. \tag{3}$$

This transformation of Kepler's equation is due to H. Andoyer,* and is remarkable because it may be used also when e approaches 1. When t is given, the determination of σ is made by successive approximations, for which H. Andoyer,† and later M. T. Subbotin,‡ using a slightly modified method, have given tables for the coefficients of σ and σ^3. In both cases the tables are constructed for logarithmic calculation.

When a calculating machine is used, and especially when we do not aim at very great accuracy, it will in most cases be profitable to leave the method of successive approximations and through development in series to change over to direct calculation.

From (2) we find

$$\frac{k}{\sqrt{2}} P = \sigma\sqrt{1-\epsilon\sigma^2} + \frac{1}{2\epsilon}\left[\frac{\arcsin\sqrt{\epsilon}\sigma}{\sqrt{\epsilon}} - \sigma\sqrt{1-\epsilon\sigma^2}\right]$$
$$= \sigma\left[1 - \frac{\sigma^2}{2}\epsilon - \frac{\sigma^4}{8}\epsilon^2 - \frac{\sigma^6}{16}\epsilon^3 - \ldots\right] + \frac{1}{2\epsilon}\left[\sigma + \frac{\sigma^3}{6}\epsilon + \frac{3\sigma^5}{40}\epsilon^2 + \frac{5\sigma^7}{112}\epsilon^3\right.$$
$$\left. + \frac{35\sigma^9}{1152}\epsilon^4 + \ldots - \sigma + \frac{\sigma^3}{2}\epsilon + \frac{\sigma^5}{8}\epsilon^2 + \frac{\sigma^7}{16}\epsilon^3 + \frac{5\sigma^9}{128}\epsilon^4 + \ldots\right];$$

$$\frac{k}{\sqrt{2}} P = \sigma + \tfrac{1}{3}\sigma^3 - (\tfrac{1}{2}\sigma^3 - \tfrac{1}{10}\sigma^5)\epsilon - (\tfrac{1}{8}\sigma^5 - \tfrac{3}{56}\sigma^7)\epsilon^2 - (\tfrac{1}{16}\sigma^7 - \tfrac{5}{144}\sigma^9)\epsilon^3 - \ldots \tag{4}$$

If we put $\tau = \tan\frac{v}{2}$ for the place in the *parabola* corresponding to P we have Barker's equation

$$\frac{k}{\sqrt{2}} P = \tau + \tfrac{1}{3}\tau^3,$$

to which (4) is reduced for $\epsilon = 0$.

If we now write

$$\sigma = \tau + k_1\epsilon + k_2\epsilon^2 + k_3\epsilon^3 + \ldots \tag{5}$$

and substitute this expression in (4), the quantities k_1, k_2, k_3 . . . may be determined as functions of τ. The result is

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* *Bull. astr.*, 35, 5, 1918.
† *Loc. cit.*
‡ *A.N.* 5606, 1928 and *Publ. Tashkent*, 2, 1929.

$$k_1 = \frac{\tau^3}{1+\tau^2}\left(\frac{1}{2} - \frac{1}{10}\tau^2\right)$$

$$k_2 = \frac{\tau^5}{(1+\tau^2)^3}\left(\frac{7}{8} + \frac{83}{280}\tau^2 - \frac{13}{56}\tau^4 - \frac{19}{1400}\tau^6\right)$$

$$k_3 = \frac{\tau^7}{(1+\tau^2)^5}\left(\frac{33}{16} + \frac{1669}{1008}\tau^2 - \frac{6001}{12600}\tau^4 - \frac{2321}{4200}\tau^6 - \frac{6793}{126000}\tau^8 - \frac{71}{25200}\tau^{10}\right);$$

. . . . . .

As τ is a function of P the quantities k (as τ itself) may be tabulated with P as argument. The calculation then proceeds in the following way.

1. With P as argument τ is determined (tables by B. Strömgren,* M. T. Subbotin † and Jens P. Möller ‡) and, using the tables given at the end of this paper, we also determine as many of the quantities k as are necessary with the given ϵ to attain the accuracy demanded. As will be shown later, we shall at the most have to use k_1 and k_2 in order to obtain 5 correct decimals. The tables are mainly constructed for computation of search-ephemerides, where a 5-figure computation gives more than sufficient accuracy, and in most actually occurring cases it will even be sufficient to use k_1 only. But by including the two higher terms the tables may often be used in 6- or 7-figure computations, as of course the attainable accuracy is further increased when e approaches 1.

2. σ is computed using (5).

3. Then we compute the quantities

$$1-\sigma^2 \quad \text{and} \quad \sigma\sqrt{1-\epsilon\sigma^2}.$$

(The square root may often with sufficient accuracy be put $= 1 - \epsilon\sigma^2/2$.)

4. The heliocentric rectangular co-ordinates can then be determined from

$$\left.\begin{aligned} x &= qP_x(1-\sigma^2) + 2qQ_x\sqrt{1-\epsilon}\;\sigma\sqrt{1-\epsilon\sigma^2} \\ y &= qP_y(1-\sigma^2) + 2qQ_y\sqrt{1-\epsilon}\;\sigma\sqrt{1-\epsilon\sigma} \\ z &= qP_z(1-\sigma^2) + 2qQ_z\sqrt{1-\epsilon}\;\sigma\sqrt{1-\epsilon\sigma^2} \end{aligned}\right\} \qquad (6)$$

where the constant factors q and $2q\sqrt{1-\epsilon}$ have previously been multiplied by the vector elements P_x, P_y, P_z and Q_x, Q_y, Q_z respectively, thus forming a new sort of Gaussian constants.

The quantities r, v and E have thus disappeared from the formulæ in a still higher degree than is the case in the formulæ adapted for calculating machine for parabola and ellipse with smaller eccentricity. If for some other purpose we should want to compute some of these quantities we give the following formulæ :—

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* *Mem. B.A.A.*, **27**, pt. 2 and *Publ. Copenhagen*, No. 58, 1927.

† *Publ. Tashkent*, **2**, 1929.

‡ *Vid. Selskabs mat.-fys. Meddelelser*, 11, 8 (Copenhagen) and *Publ. Copenhagen*, No. 82, 1932.

$$\left.\begin{aligned} r = q(1+e\sigma^2)\,; \quad \cos v = \frac{1-\sigma^2}{1+e\sigma^2}\,; \quad \sin v = \frac{2\sqrt{1-\epsilon}\;\sigma\;\sqrt{1-\epsilon\sigma^2}}{1+e\sigma^2}\,; \\ \tan\frac{v}{2} = \sqrt{\frac{1-\epsilon}{1-\epsilon\sigma^2}}\,\sigma\,; \quad \cos E = 1-2\epsilon\sigma^2\,; \quad \sin E = 2\sqrt{\epsilon}\;\sigma\;\sqrt{1-\epsilon\sigma^2}\,; \\ \sigma^2 = \frac{\sin^2\frac{E}{2}}{\epsilon} = \frac{1-\cos E}{1-e} = \frac{1-\cos v}{1+e\cos v} = \frac{\tan^2\frac{v}{2}}{1-\epsilon\left(1-\tan^2\frac{v}{2}\right)}. \end{aligned}\right\} \quad (7)$$

In the following we shall examine the accuracy attainable by the given method and by the usual method of solution of Kepler's equation, so as to determine the limit between those cases where the one or the other method is to be preferred.

A convenient method of solving Kepler's equation for eccentricities not in the vicinity of 1 is as follows: The equation is written $E = M + e\sin E$. M is placed in the product register and e on the keyboard. Then, with the aid of a trigonometrical table in degrees and decimals (or in radians), the machine is operated till the product register shows a figure E, the sine of which is equal to the figure in the multiplier register. When the desired accordance is attained, $\sin E$ and the corresponding $\cos E$ are written down.*

If e is near 1 this method becomes unsatisfactory, because the equation is satisfied to the given degree of accuracy for a series of values of $\sin E$. To determine the magnitude of the error Δ in $\sin E$ we write Kepler's equation

$$E + e\Delta = M + e(\sin E + \Delta).$$

The difference between the two quantities $\sin(E+e\Delta)$ and $\sin E + \Delta$ may in unfavourable cases rise to $0{\cdot}5 \times 10^{-n}$, where n is the number of decimals employed. The maximum value of Δ is thus determined from

$$\begin{aligned} 0{\cdot}5 \times 10^{-n} &= \sin E + \Delta - \sin(E+e\Delta) \\ &= \sin E + \Delta - (\sin E + e\Delta\cos E)\,; \end{aligned}$$

hence

$$\Delta = \frac{0{\cdot}5 \times 10^{-n}}{1 - e\cos E}.$$

The quantity

$$K = \frac{1}{1-e\cos E}$$

is thus decisive for the accuracy with which $\sin E$ can be found from a given trigonometrical table, as the maximum error in $\sin E$ is K times the maximum error of the table. If, for instance, $e = 0{\cdot}9$ and if we examine the most un-

* This method does not seem to be generally known. It is described by B. Noumerov in *Bull. de l'Institut Astronomique, Leningrad*, No. 12, 1926, but it is known to the writer that L. J. Comrie used it some years before.

favourable place in the orbit, *i.e.* in the vicinity of perihelion, we find $K = 10$, and sin E will have an average error of 2·5 units in the last decimal. If the calculation of the ephemeris is to 5 figures, and if we want the fifth decimal of sin E to be correct, we have to use a 6-figure table for the solution of Kepler's equation.

We shall adopt $K = 10$ as the practical limit beyond which the usual method for the solution of Kepler's equation is no longer appropriate.

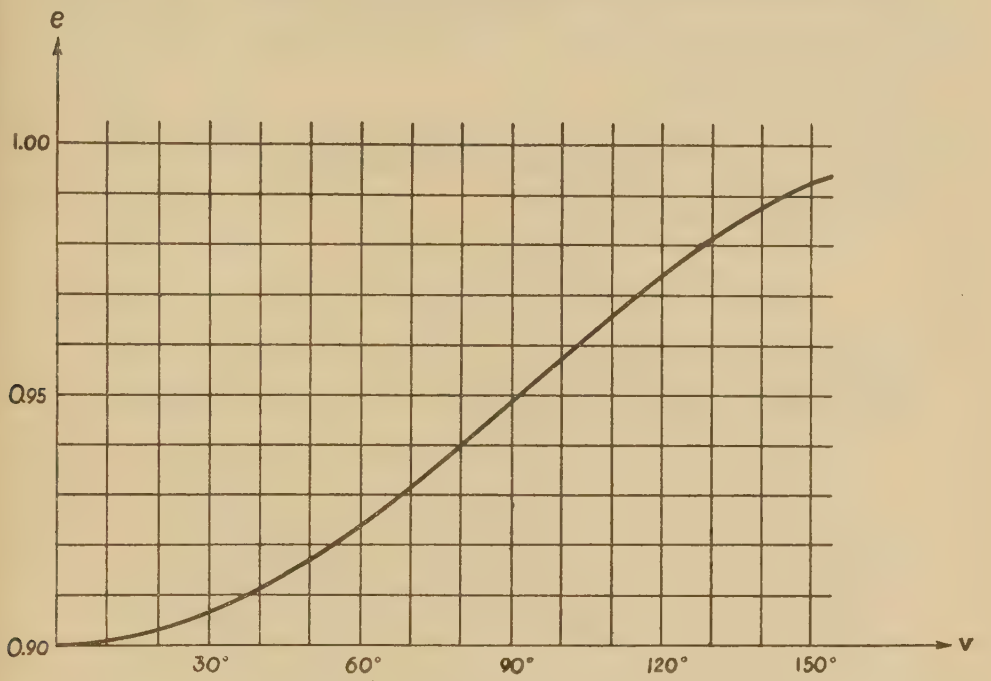

FIG. 1.—e *as a Function of* v *in the Limiting Case.*

A modification of the above method, suggested by B. Strömgren, can be used with profit, especially in the limiting cases near $K = 10$. Kepler's equation is written in the form: $\sin E = \frac{1}{e} E - \frac{1}{e} M$, and $\frac{1}{e}$ is placed on the keyboard and $-\frac{1}{e} M$ in the product register. E then appears in the multiplier register and sin E in the product register. The advantage obtained is the avoidance of a number of interpolations, because the argument E is built up step by step in the multiplier register and does not, as in the other case, appear immediately with the full number of decimals.

For $K > 10$ we have to leave the usual methods and turn to the special methods for nearly parabolic orbits. As mentioned above this limit is, near perihelion, reached when $e = 0{\cdot}9$, but at greater distances from perihelion the limit is reached at greater values of e. The equation

$$K = \frac{1}{1 - e \cos E} = 10 \qquad (8)$$

gives e as a function of E in the limiting case. A more lucid representation is obtained by introducing v or P instead of E. We find

$$\frac{1}{1-e\cos E}=\frac{\cos v}{\cos E-e}=10.$$

By eliminating $\cos E$ from these two equations we get

$$10e^2+e\cos v-9=0, \tag{9}$$

which gives e as a function of v.

Furthermore, we have

$$P=\frac{t-T}{q^{3/2}}=\frac{M}{k(1-e)^{3/2}}=\frac{E-e\sin E}{k(1-e)^{3/2}}$$

which, together with (8), gives

$$P=\frac{\operatorname{arc}\cos\dfrac{9}{10e}-\sqrt{e^2-0{\cdot}81}}{k(1-e)^{3/2}}. \tag{10}$$

This equation gives e implicitly as a function of P.

Figs. 1 and 2 are graphic representations of (9) and (10). Fig. 1 shows,

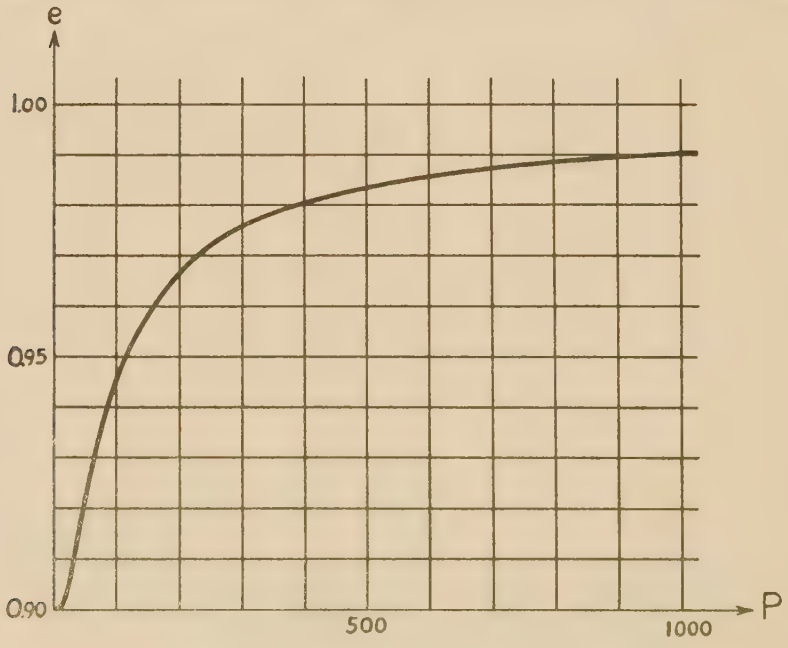

Fig. 2.—e *as a Function of* P *in the Limiting Case.*

for instance, that for $v=145^\circ$ only values of e greater than 0·99 will need the use of the special methods for nearly parabolic orbits. Fig. 2 shows that for $e=0{\cdot}99$ these methods must be used until about $P=1000$; thus, for example, for $q=1$ the special methods are required for about three

years on both sides of perihelion. In this way it is possible, without previous experiments, to decide immediately whether a given case has to be treated in one way or in the other.

As mentioned already the tables of k_1, k_2 and k_3 for the above developed treatment of nearly parabolic orbits are meant for 5-figure computation. As seen above the maximum values (reached in the beginning of the tables) of the factors ϵ and ϵ^2 of k_1 and k_2 are $\frac{1}{20}$ and $\frac{1}{400}$, so that k_1 and k_2 are given with 4 and 3 decimals respectively. In other parts of the tables, where the maximum value ϵ_m of ϵ is still smaller, k_1 and k_2 are given with fewer decimals, but $k_1\epsilon_m$ and $k_2\epsilon_m{}^2$ can everywhere be determined with 5 correct decimals.

Fig. 3 gives the numerical value of $k_3\epsilon_m{}^3$ in units of the sixth decimal. It will be seen that this term is always negligible in a 5-figure computation,

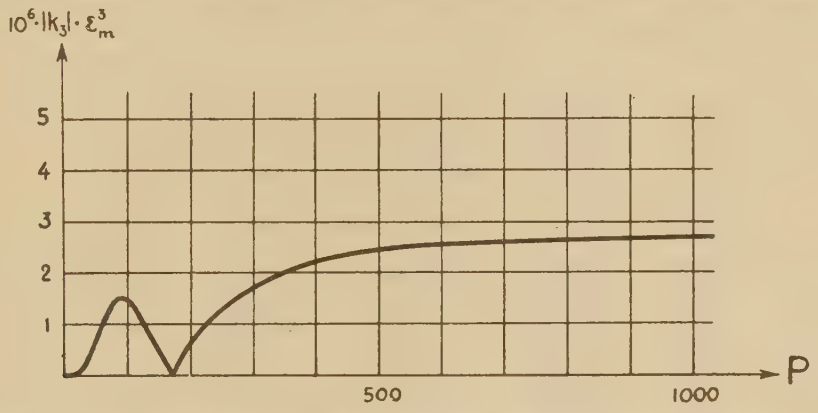

FIG. 3.—*Maximum Numerical Value of the Third Term in the Series for σ in Units of the Sixth Decimal.*

so that the use of k_1 and k_2 only insures 5 correct decimals. It is evident that the use of k_2 will often be superfluous and that the tables will in many cases be able to supply more than 5 decimals.

Attention is drawn to the fact that the method may be used unaltered for nearly parabolic *hyperbolic* orbits—all formulæ are directly valid and ϵ is thus negative.

Illustration.—In the classical illustration from Gauss's *Theoria Motus* we get, by rounding-off to 5 figures,

$$e = 0{\cdot}96765 \qquad q^{3/2} = 0{\cdot}44512$$
$$q = 0{\cdot}58298$$
$$t - T = 63{\cdot}544 \qquad P = 142{\cdot}757.$$

The value of P shows (fig. 2) that the illustration is just within the limit of the nearly parabolic domain in the sense explained above, and it is thus relatively unfavourable for the use of the tables. We find

$$\tau = 1{\cdot}18366$$
$$k_1 = 0{\cdot}2485 \qquad \epsilon = 0{\cdot}01618 \qquad \sigma = 1{\cdot}18772 \qquad \sqrt{1-\epsilon\sigma^2} = 0{\cdot}98852$$
$$k_2 = 0{\cdot}134 \qquad \epsilon^2 = 0{\cdot}00026 \qquad \sigma^2 = 1{\cdot}41068 \qquad \sigma\sqrt{1-\epsilon\sigma^2} = 1{\cdot}17408.$$

In a calculation of an ephemeris we then have to compute the heliocentric co-ordinates using the formulæ (6).

For comparison with the results in the *Theoria Motus* we determine r and v by (7) and get

$$1+e\sigma^2 = 2{\cdot}36504 \qquad \cos v = -0{\cdot}17365$$
$$r = 1{\cdot}37877 \qquad v = 100^\circ\ 0'\ 0''.$$

It will be seen that a discordance of one unit in the fifth decimal is found in r, while v is in complete accordance. Corresponding computations may be done with 6 decimals, and the maximum disagreement in the sixth decimal of the result will be one unit.

The formulæ necessary for the use of the table are

$$P=(t-T)\,q^{-\frac{3}{2}}; \qquad \epsilon=\frac{1-e}{2}; \qquad \sigma=\tau+k_1\epsilon+k_2\epsilon^2+k_3\epsilon^3+\ldots$$

$$x=qP_x\,(1-\sigma^2)+2qQ_x\sqrt{1-\epsilon}\;\sigma\sqrt{1-\epsilon\sigma^2}$$
$$y=qP_y\,(1-\sigma^2)+2qQ_y\sqrt{1-\epsilon}\;\sigma\sqrt{1-\epsilon\sigma^2}$$
$$z=qP_z\,(1-\sigma^2)+2qQ_z\sqrt{1-\epsilon}\;\sigma\sqrt{1-\epsilon\sigma^2}.$$

P	k_1	k_2	k_3	P	k_1	k_2	k_3
0	+0·0000	0·000	0·00	25	+0·0117	+0·002	0·00
1	·0000	·000		26	·0129	·002	
2	·0000	·000		27	·0143	·002	
3	·0000	·000		28	·0157	·003	
4	·0001	·000		29	·0172	·003	
5	+0·0001	0·000	0·00	30	+0·0187	+0·003	0·00
6	·0002	·000		31	·0203	·004	
7	·0003	·000		32	·0220	·004	
8	·0005	·000		33	·0237	·005	
9	·0006	·000		34	·0255	·006	
10	+0·0009	0·000	0·00	35	+0·0274	+0·006	0·00
11	·0012	·000		36	·0293	·007	
12	·0015	·000		37	·0312	·008	
13	·0019	·000		38	·0333	·009	
14	·0023	·000		39	·0353	·009	
15	+0·0028	0·000	0·00	40	+0·0374	+0·010	0·00
16	·0034	·000		41	·0396	·011	
17	·0040	·000		42	·0418	·012	
18	·0047	·000		43	·0440	·013	
19	·0055	·000		44	·0463	·014	
20	+0·0064	+0·001	0·00	45	+0·0486	+0·015	+0·01
21	·0073	·001		46	·0509	·017	
22	·0083	·001		47	·0533	·018	
23	·0093	·001		48	·0557	·019	
24	·0105	·001		49	·0581	·020	
25	+0·0117	+0·002	0·00	50	+0·0606	+0·022	+0·01

P	k_1	k_2	k_3	P	k_1	k_2	k_3
50	+0·0606	+0·022	+0·01	95	+0·1710	+0·097	+0·07
51	·0630	·023		96	·1731	·098	
52	·0655	·025		97	·1752	·100	
53	·0680	·026		98	·1773	·101	
54	·0705	·028		99	·1794	·102	
55	+0·0731	+0·029	+0·02	100	+0·1814	+0·104	+0·07
56	·0756	·031		101	·1835	·105	
57	·0782	·032		102	·1855	·107	
58	·0807	·034		103	·1874	·108	
59	·0833	·035		104	·1894	·109	
60	+0·0859	+0·037	+0·02	105	+0·1913	+0·110	+0·08
61	·0885	·039		106	·1933	·112	
62	·0911	·040		107	·1952	·113	
63	·0936	·042		108	·1970	·114	
64	·0962	·044		109	·1989	·115	
65	+0·0988	+0·045	+0·03	110	+0·2007	+0·116	+0·08
66	·1014	·047		111	·2025	·117	
67	·1040	·049		112	·2043	·118	
68	·1065	·051		113	·2060	·119	
69	·1091	·053		114	·2078	·120	
70	+0·1116	+0·054	+0·03	115	+0·2095	+0·121	+0·08
71	·1142	·056		116	·2112	·122	
72	·1167	·058		117	·2129	·123	
73	·1193	·060		118	·2145	·124	
74	·1218	·061		119	·2161	·125	
75	+0·1243	+0·063	+0·04	120	+0·2178	+0·125	+0·09
76	·1268	·065		121	·2193	·126	
77	·1293	·067		122	·2209	·127	
78	·1317	·069		123	·2224	·127	
79	·1342	·070		124	·2240	·128	
80	+0·1366	+0·072	+0·05	125	+0·2255	+0·129	+0·08
81	·1390	·074		126	·2269	·129	
82	·1414	·076		127	·2284	·130	
83	·1438	·077		128	·2298	·130	
84	·1462	·079		129	·2312	·131	
85	+0·1485	+0·081	+0·06	130	+0·2326	+0·131	+0·08
86	·1509	·082		131	·2340	·132	
87	·1532	·084		132	·2353	·132	
88	·1555	·086		133	·2367	·132	
89	·1577	·087		134	·2380	·133	
90	+0·1600	+0·089	+0·06	135	+0·2393	+0·133	+0·08
91	·1622	·091		136	·2405	·133	
92	·1645	·092		137	·2418	·133	
93	·1667	·094		138	·2430	·134	
94	·1688	·095		139	·2442	·134	
95	+0·1710	+0·097	+0·07	140	+0·2454	+0·134	+0·07

P	k_1	k_2	k_3	P	k_1	k_2	k_3
140	+0·2454	+0·134	+0·07	185	+0·2799	+0·104	−0·05
141	·2466	·134		186	·2803	·102	
142	·2477	·134		187	·2807	·101	
143	·2488	·134		188	·2810	·099	
144	·2500	·134		189	·2814	·098	
145	+0·2510	+0·134	+0·07	190	+0·2817	+0·096	−0·07
146	·2521	·134		191	·2820	·095	
147	·2532	·133		192	·2823	·093	
148	·2542	·133		193	·2826	·092	
149	·2552	·133		194	·2829	·090	
150	+0·2562	+0·133	+0·06	195	+0·2832	+0·088	−0·09
151	·2572	·133		196	·2834	·086	
152	·2581	·132		197	·2836	·085	
153	·2591	·132		198	·2839	·083	
154	·2600	·132		199	·2841	·081	
155	+0·2609	+0·131	+0·05	200	+0·2843	+0·079	−0·12
156	·2618	·131		201	·2844	·078	
157	·2626	·130		202	·2846	·076	
158	·2635	·130		203	·2848	·074	
159	·2643	·129		204	·2849	·072	
160	+0·2651	+0·129	+0·04	205	+0·2850	+0·070	−0·15
161	·2659	·128		206	·2851	·068	
162	·2667	·127		207	·2852	·066	
163	·2674	·127		208	·2853	·064	
164	·2682	·126		209	·2854	·062	
165	+0·2689	+0·125	+0·02	210	+0·2855	+0·060	−0·18
166	·2696	·124		211	·2855	·058	
167	·2703	·124		212	·2855	·055	
168	·2710	·123		213	·2856	·053	
169	·2716	·122		214	·2856	·051	
170	+0·2722	+0·121	+0·01	215	+0·2856	+0·049	−0·21
171	·2729	·120		216	·2856	·046	
172	·2735	·119		217	·2856	·044	
173	·2741	·118		218	·2855	·042	
174	·2746	·117		219	·2855	·040	
175	+0·2752	+0·116	−0·01	220	+0·2854	+0·037	−0·24
176	·2757	·115·		221	·2853	·035	
177	·2763	·114		222	·2853	·032	
178	·2768	·113		223	·2852	·030	
179	·2773	·111		224	·2851	·027	
180	+0·2778	+0·110	−0·03	225	+0·2849	+0·025	−0·27
181	·2782	·109		226	·2848	·022	
182	·2787	·108		227	·2847	·020	
183	·2791	·106		228	·2845	·017	
184	·2795	·105		229	·2844	·015	
185	+0·2799	+0·104	−0·05	230	+0·2842	+0·012	−0·31

P	k_1	k_2	k_3	P	k_1	k_2	k_3
230	+0·2842	+0·012	−0·31	275	+0·2661	−0·131	−0·69
231	·2840	·009		276	·2655	·135	
232	·2838	·007		277	·2649	·139	
233	·2836	·004		278	·2643	·142	
234	·2834	+0·001		279	·2636	·146	
235	+0·2832	−0·001	−0·34	280	+0·2630	−0·150	−0·74
236	·2829	·004		281	·2623	·154	
237	·2827	·007		282	·2617	·158	
238	·2824	·010		283	·2610	·161	
239	·2822	·013		284	·2603	·165	
240	+0·2819	−0·016	−0·38	285	+0·2597	−0·169	−0·79
241	·2816	·019		286	·2590	·173	
242	·2813	·021		287	·2583	·177	
243	·2810	·024		288	·2576	·181	
244	·2807	·027		289	·2569	·185	
245	+0·2803	−0·030	−0·42	290	+0·2561	−0·189	−0·84
246	·2800	·033		291	·2554	·193	
247	·2797	·036		292	·2547	·197	
248	·2793	·040		293	·2539	·201	
249	·2789	·043		294	·2532	·205	
250	+0·2785	−0·046	−0·46	295	+0·2524	−0·209	−0·90
251	·2782	·049		296	·2517	·213	
252	·2778	·052		297	·2509	·218	
253	·2774	·055		298	·2501	·222	
254	·2769	·058		299	·2493	·226	
255	+0·2765	−0·062	−0·50	300	+0·2485	−0·230	−0·95
256	·2761	·065					
257	·2756	·068		300	+0·249	−0·23	−1
258	·2752	·072		310	·240	·27	
259	·2747	·075		320	·231	·32	
260	+0·2742	−0·078	−0·55	330	·222	·37	
261	·2738	·082		340	·211	·41	
262	·2733	·085		350	+0·200	−0·46	−2
263	·2728	·088		360	·189	·52	
264	·2723	·092		370	·177	·57	
265	+0·2718	−0·095	−0·59	380	·165	·63	
266	·2712	·099		390	·152	·68	
267	·2707	·102		400	+0·138	−0·74	−2
268	·2702	·106		410	·125	·80	
269	·2696	·109		420	·110	·86	
270	+0·2690	−0·113	−0·64	430	·096	·92	
271	·2685	·117		440	·081	0·99	
272	·2679	·120		450	+0·065	−1·05	−3
273	·2673	·124		460	·050	·12	
274	·2667	·128		470	·034	·19	
275	+0·2661	−0·131	−0·69	480	+0·017	·26	
				490	·000	·33	
				500	−0·017	−1·40	−4

P	k_1	k_2	k_3	P	k_1	k_2	k_3
500	−0·017	−1·40	−4	750	−0·518	−3·6	−11
510	·034	·47		760	·541	3·7	
520	·052	·55		770	·563	3·8	
530	·070	·62		780	·586	3·9	
540	·088	·70		790	·609	4·0	
550	−0·106	−1·78	−5	800	−0·632	−4·1	−13
560	·125	1·86		810	·655	4·2	
570	·144	1·94		820	·678	4·3	
580	·163	2·02		830	·701	4·4	
590	·182	2·10		840	·725	4·5	
600	−0·202	−2·19	−7	850	−0·748	−4·6	−15
610	·222	2·27		860	·772	4·7	
620	·242	2·36		870	·796	4·8	
630	·262	2·44		880	·819	4·9	
640	·282	2·53		890	·843	5·0	
650	−0·303	−2·62	−8	900	−0·867	−5·1	−17
660	·324	2·71		910	·892	5·2	
670	·345	2·80		920	·916	5·3	
680	·366	2·89		930	·940	5·5	
690	·387	2·98		940	·965	5·6	
700	−0·409	−3·1	−10	950	−0·989	−5·7	−19
710	·430	3·2		960	1·014	5·8	
720	·452	3·3		970	·039	5·9	
730	·474	3·4		980	·064	6·0	
740	·496	3·5		990	·089	6·2	
750	−0·518	−3·6	−11	1000	−1·114	−6·3	−21

University Observatory,
Copenhagen.

RT HYDRÆ.

P. M. Ryves.

1. Variability to the extent of two magnitudes (8·0 to 10·1 photographic) was discovered in the star B.D. −5° 2550 by Mrs. Fleming in 1898. The position for 1855 is R.A. $8^h\ 22^m\ 32^s$, dec. S. 5° 50′ 15″; approximate position for 1925 R.A. $8^h\ 25^m{\cdot}9$, dec. S. 5° 52′; spectrum M*d* (or M6) with emission lines; designation 082405, RT Hydræ.

The second catalogue of H.C.O. (1907) gave the visual range as 7·4 to 9·8, adding the note "The measures of Harvard photographs from 1888 March 18 to 1901 December 25 appear to show evidence of a period of about 280 days. . . . The observations are too scattering, however, to determine a precise value for the period and it may be irregular."

2. Observations were commenced at Zaragoza in 1910 as part of the programme of study of difficult stars of small range. The present note is based on 348 observations made in the years 1910–14 and 1923–33. A refracting telescope with apertures of 2 inches (stop) and 3 inches is used, except near minimum, when a 5-inch refractor is employed.

The variable was identified by means of a Harvard photograph kindly supplied by the late Professor E. C. Pickering. The magnitudes of some comparison stars are marked on the photograph and these stars were chiefly used during the first years. When Father Hagen's chart (A.S.V., Series IV) was received it was found that there were some discrepancies which in the case of No. 5 amounts to half a magnitude : Hagen 7·3, Harvard 7·8.

The light-scale has been deduced from the estimates and adjusted to the Harvard photometric scale in the usual way. In assigning the magnitudes adopted, account has been taken of Hagen's grade estimates, and greater weight has been given to these values in the case of stars observed here on comparatively few nights.

The comparison stars most frequently used are shown in the adjoined table :

TABLE

B.D.	Hagen No.	Hagen Mag.	H.C.O. Mag.	Adopted Prov. Mag.	
−5°2544	6	7·4	7·44	7·3	
−5 2529	8	7·5	...	7·7	
−5 2573	13	8·2	...	8·0	
−5 2538	16	8·3	8·52	8·5	
−5 2535	17	8·5	...	8·7	
−5 2547	18	8·6	8·62	8·6	
−5 2563	20	8·9	...	8·9	
−6 2604	23	9·1	...	8·8	Dbl., combined mag.
−5 2557	26	9·5	...	9·3	

It is intended to revise the sequence at some future time, after which the estimates will be re-reduced and, if possible, published in tabular form. In the meantime the observations are shown on the accompanying diagram and can be read off with reasonable accuracy by means of a scale graduated in millimetres.

On account of its position in the sky RT Hydræ is unobservable from this station during about four months in each year.

3. The following are the principal results obtained so far :—

Amplitude.—The extreme visual range observed is two magnitudes, 7·4 to 9·4, but in any given cycle it is generally less, occasionally as little as one magnitude. On account of the small light grasp of the telescope used at maximum the above quantities are practically free from Purkinje effect.

Period.—A mean period of $252^{d}.5$ is indicated. There are fluctuations in the length of the period, but the material is not sufficient to determine whether they are systematic. Elements :—

$$\text{Max.} = \text{J.D. } 2418919 + 252^{d}.5\ C.$$

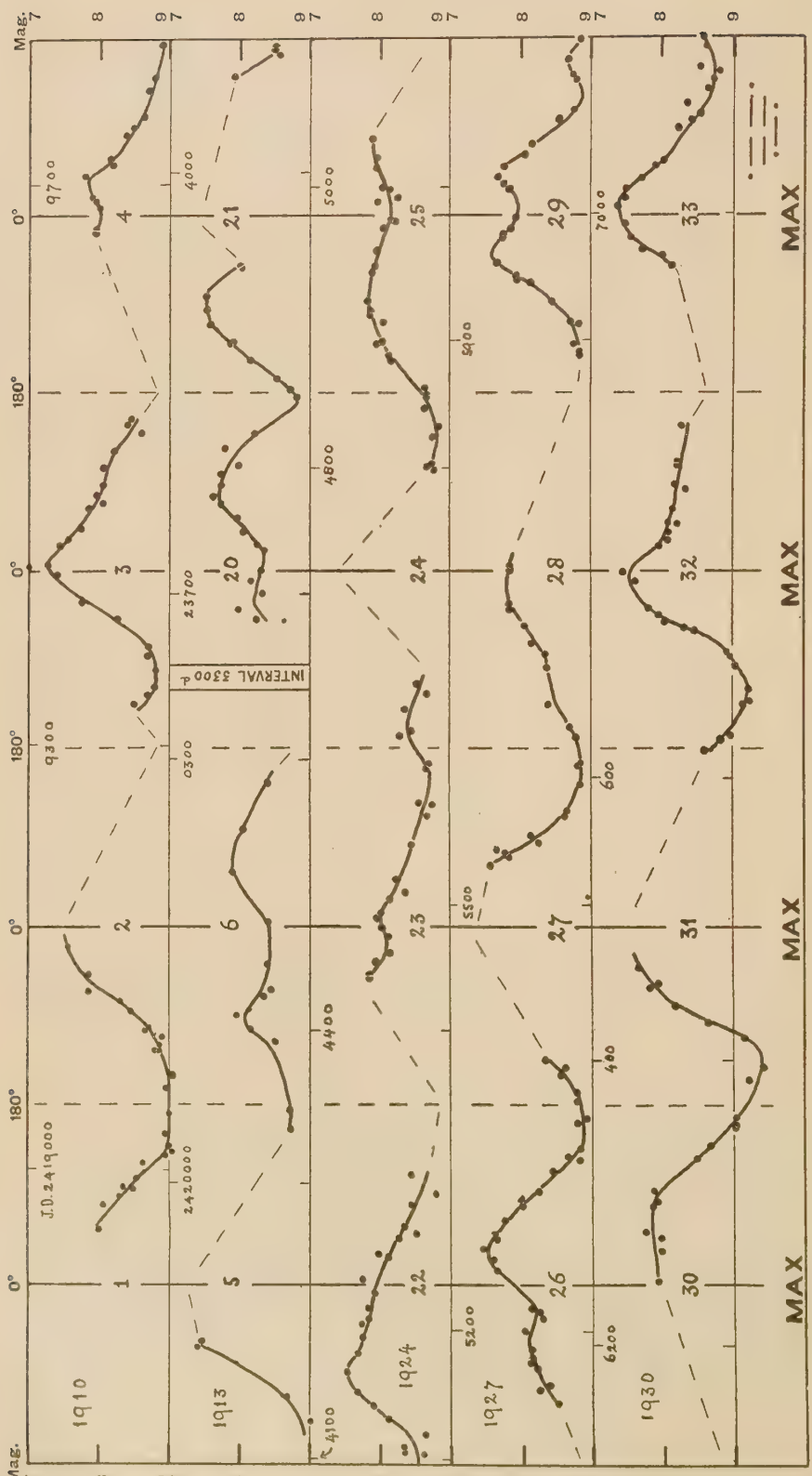

FIG. 1.—*RT Hydræ. Scale: 1 mag. = 50 days = 1 centimetre.*
The width of the diagram is equivalent to 1010 days, i.e. 4 cycles of the variable. Points of reference are provided at convenient Julian days. These may be joined by a pencil line, and the date of any observation read off to within a day or two by means of a millimetre scale.

$M - m$ is about half the period, but seems to be somewhat less before a narrow maximum and more before a broad one.

Character of Light-changes.—The variation may be described as of the *long-period* type, but with irregularities, the light-curve being in many respects similar to that of RW Sagittarii.* Two distinct types of maximum can be recognised, one comparatively narrow and generally bright, the other broader and fainter. The broad maxima frequently show a central depression dividing them into two peaks. Sometimes the maximum is very long and the peaks widely separated (cycles 6, 20, 25) and at such times the light-curve bears a resemblance to that of a star of the RV Tauri type. I think, however, that the resemblance is more apparent than real as the feature is not permanent. The double maxima are sometimes narrower as in cycle 29, which has two well-defined peaks, 60 days apart, recalling

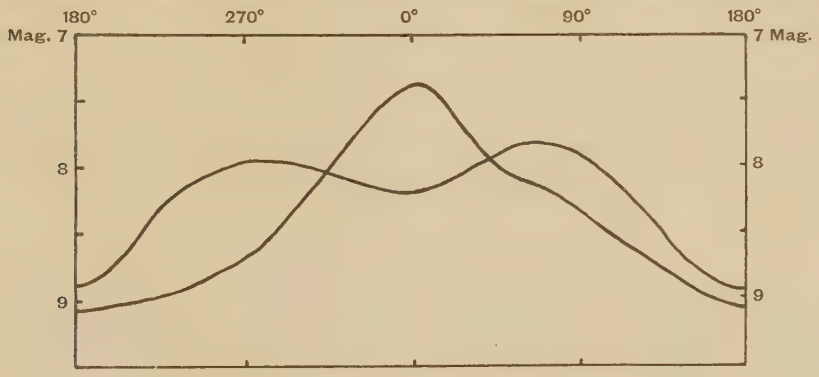

FIG. 2.—*RT Hydræ. Different types of maximum.*

RR Centauri. The depression or secondary minimum sometimes occurs several weeks before or after the maximum phase, causing an inflexion or "hump" on the ascending or descending branch of the curve, a well-known characteristic of many mira-type variables (cycles 22, 26). Lastly, the narrow maxima closely resemble those of typical long-period stars free from inflexions about that phase, though there may be irregularities near minimum. Cycles 3 (1912) and 33 (1933) are good examples. At the minimum 33–34, which extends beyond the diagram, there is a remarkable recovery or secondary maximum, and a slight indication of a similar tendency can be seen at minimum 1–2 (1910). An attempt to discover a periodicity in the secondary fluctuations has not been successful.

The variation of RT Hydræ is sometimes comparatively regular over several successive cycles, whilst at other times irregularities persist for a considerable period. In 1924–1926 the star was evidently passing through a disturbed period; the maxima did not occur at all exactly at the times indicated by the elements, though the minima were more punctual.

4. The peculiarities of the light-curve of RT Hydræ, described in the previous paragraphs, may be explained on the supposition of a secondary variation superimposed on a regular periodic light-change. Such secondary

* P. M. Ryves, *M.N.*, **84**, 34, 1923.

variation would appear to be irregular but may conform to some obscure periodicity. A superimposed variation of an amplitude such as would cause only a slight modification of the light-curve of a typical mira-type star would be sufficient to produce considerable changes in the shape of the curve of a star such as RT Hydræ on account of the smaller amplitude of the main periodic variation.

The above conclusions are based on work done here; the writer has had no opportunity of utilising observations made elsewhere. Prager, in his *Katalog und Ephemeriden*, has assumed a variation of RV Tauri type and made predictions which were not strictly fulfilled, and the latest Harvard catalogue gives the period as $254^{d}.8$, but no epoch.

Zaragoza, Spain:
1933 *June* 5.

MEAN AREAS AND HELIOGRAPHIC LATITUDES OF SUNSPOTS IN THE YEAR 1932.

(Communicated by the Astronomer Royal)

The results given here are in continuation of those in *M.N.*, **92**, 742, 1932, and are derived from the measurement of photographs taken at the Royal Observatories of Greenwich and the Cape, and at the Kodaikanal Observatory, India.

TABLE I

No. of Rotation	Rotation Commenced G.C.T.	No. of Days on which Sun was photographed	Mean Daily Areas					
			Projected *			Corrected for Foreshortening †		
			Umbræ	Whole Spots	Faculæ	Umbræ	Whole Spots	Faculæ
1047	Dec. 21·85	27	31	136	549	21	93	622
1048	Jan. 18·19	28	36	170	392	28	131	456
1049	Feb. 14·53	27	67	335	338	48	243	405
1050	Mar. 12·86	27	14	53	255	10	40	308
1051	Apr. 9·16	28	72	327	179	53	242	235
1052	May 6·41	27	102	503	424	75	367	507
1053	June 2·62	27	107	507	690	76	363	837
1054	29·82	27	34	138	498	23	98	563
1055	July 27·02	27	26	135	196	20	107	230
1056	Aug. 23·25	28	11	42	153	8	31	186
1057	Sept. 19·51	27	5	17	212	4	15	256
1058	Oct. 16·80	27	22	111	248	16	80	289
1059	Nov. 13·10	28	51	250	409	51	241	475
1060	Dec. 10·41	27	63	300	375	50	233	485

* Expressed in millionths of the Sun's disc.
† Expressed in millionths of the Sun's hemisphere.

Table I gives the mean daily areas of umbræ, whole spots and faculæ for each synodic rotation of the Sun included in the year 1932, and the mean values for each year as a whole from 1923 to 1932 are given in Table II.

TABLE II

Year	No. of Days		Mean Daily Areas					
			Projected *			Corrected for Foreshortening †		
	Photographed	Without Spots	Umbræ	Whole Spots	Faculæ	Umbræ	Whole Spots	Faculæ
1923	365	171	15	74	189	10	55	222
1924	366	97	62	361	498	46	276	575
1925	365	19	189	1102	1508	141	830	1750
1926	365	2	284	1651	2159	217	1262	2556
1927	365	0	248	1417	1812	186	1058	2212
1928	366	0	333	1884	2134	244	1390	2589
1929	365	0	334	1695	2190	245	1242	2567
1930	365	4	145	705	1378	106	516	1630
1931	365	42	79	372	679	57	275	801
1932	366	92	46	216	335	35	163	400

TABLE III

No. of Rotation	Rotation Commenced G.C.T.	Spots North of the Equator		Spots South of the Equator		Mean Latitude of entire Spotted Area	Mean Distance from Equator of all Spots
		Mean Daily Area	Mean Heliographic Latitude	Mean Daily Area	Mean Heliographic Latitude		
			°		°	°	°
1047	Dec. 21·85	12	10·71	81	13·16	−10·05	12·84
1048	Jan. 18·19	72	12·59	59	12·56	+ 1·32	12·58
1049	Feb. 14·53	238	8·35	5	9·43	+ 8·00	8·37
1050	Mar. 12·86	27	12·77	14	8·38	+ 5·58	11·28
1051	Apr. 9·16	208	9·80	34	8·96	+ 7·16	9·68
1052	May 6·41	310	6·38	57	7·71	+ 4·19	6·59
1053	June 2·62	260	4·52	103	6·64	+ 1·36	5·12
1054	29·82	13	11·98	86	8·01	− 5·44	8·52
1055	July 27·02	1	9·29	107	8·40	− 8·31	8·40
1056	Aug. 23·25	1	9·30	30	7·13	− 6·75	7·18
1057	Sept. 19·51	9	9·93	5	4·12	+ 4·85	7·83
1058	Oct. 16·80	75	9·99	6	8·07	+ 8·71	9·85
1059	Nov. 13·10	241	9·01	0	...	+ 9·01	9·01
1060	Dec. 10·41	233	9·84	0·4	0·40	+ 9·82	9·82

Table III gives for each rotation in the year 1932 the mean daily area of the whole spots (corrected for foreshortening) and the mean heliographic latitude of the spotted areas for both northern and southern hemispheres. The mean heliographic latitude of the entire spotted area and the mean distance from the equator of all spots are also tabulated. The mean values for the

* Expressed in millionths of the Sun's disc.

† Expressed in millionths of the Sun's hemisphere.

years 1923 to 1932 are given in Table IV. Tables II and IV are in continuation of similar tables given in *Monthly Notices*: for the years 1874 to 1888—**49**, 381–382; 1889 to 1902—**63**, 465–466; 1901 to 1914—**76**, 402–403; 1913 to 1924—**85**, 1007–1008.

TABLE IV

Year	No. of Days on which Sun was photographed	Spots North of the Equator		Spots South of the Equator		Mean Latitude of entire Spotted Area	Mean Distance from Equator of all Spots
		Mean Daily Area	Mean Heliographic Latitude	Mean Daily Area	Mean Heliographic Latitude		
			°		°	°	°
1923	365	33	13·88	22	17·32	+ 1·34	15·26
1924	366	232	22·19	45	25·54	+14·49	22·73
1925	365	517	20·70	313	19·37	+ 5·60	20·20
1926	365	663	19·27	599	17·98	+ 1·60	18·66
1927	365	379	17·75	679	13·54	− 2·33	15·05
1928	366	727	11·93	663	15·23	− 1·02	13·50
1929	365	656	10·73	586	10·27	+ 0·82	10·51
1930	365	286	10·60	230	8·96	+ 1·88	9·87
1931	365	200	7·91	75	9·39	+ 3·21	8·31
1932	366	123	8·22	40	8·62	+ 4·06	8·32

The rotations in Tables I and III are numbered in continuation of Carrington's series.

The chief features of the record for the year 1932 are as follows :—

1. The progressive decline of solar activity towards the minimum of the 11-year cycle shown by (*a*) the decrease in the number of groups of sunspots, (*b*) the decrease in the mean daily area of sunspots and faculæ, (*c*) the increase in the number of days without spots.

2. On 92 days no spots were seen on the Sun's disc; on 56 days no faculæ were recorded, and on 30 days neither spots nor faculæ.

3. The northern hemisphere was the more active.

4. The mean latitude of spots is the same as for 1931. Excluding two small short-lived spots with respective latitudes of +48° (Jan. 15) and −35° (Feb. 3), the latitudes ranged from 0° to 17°.

5. The number of groups lasting for two days or longer was 71; of these, 37 were in the northern hemisphere and 34 in the southern. In addition, there were 42 very small groups (usually consisting of one or two tiny spots) measured on one day only; 25 were in the northern hemisphere and 17 in the southern.

Royal Observatory, Greenwich :
1933 *July*.

HORNSBY'S OBSERVATIONS OF URANUS, 1790–1798.

Bernard Francis Bawtree.

(Communicated by Dr. H. Knox-Shaw)

The following comparisons of Hornsby's observations of Uranus with places obtained from Newcomb's tables * are in continuation of those for the years 1781–1784, for which the tabular places were computed by Dr. Crommelin and which appear on p. 141 of *The Observations of the Reverend Thomas Hornsby*, Oxford, 1932.

Uranus

Observed *minus* Computed

Date	R.A.	Dec.	Date	R.A.	Dec.	Date	R.A.	Dec.
1790	s	″	1792	s	″	1795	s	″
Jan. 27	−·14	−0·3	Mar. 9	−·38	−0·1	Apr. 8	+·01	+ 1·5
Feb. 16	−·09	+1·4	12	−·19	−1·7	23	−·05	+ 1·9
17	+·02	+1·3	18	−·09	−2·2	26	−·06	+ 1·5
20	−·06	+1·4				29	−·18	+ 1·6
21	+·01	+0·6	1793					
26	−·11	0·0	Mar. 14	−·05	+0·8	1796		
Mar. 10	−·05	+1·6	21	−·09	+1·0	Mar. 30	−·02	+ 0·1
16	−·15	−2·6	29	−·06	+2·4	Apr. 8	−·12	− 2·4
18	+·04	+1·5	Apr. 4	−·22	−0·4	9	−·14	− 3·5
19	−·28	+0·3	5	−·02	+2·7	20	−·19	− 2·8
20	−·02	+0·6	6	−·08	+1·8	23	−·20	− 2·7
21	−·12	+2·2	7	−·10	+2·1	24	+·09	− 0·9
27	−·17	+1·0	8	−·02	0·0	25	−·19	+ 1·1
28	−·01	+0·3	10	−·07	+2·7	May 1	+·08	+ 2·6
31	−·07	+0·4	21	−·01	−3·4			
Apr. 3	−·08	+0·6	22	·00	−6·4	1798		
4	+·07	+1·5				Apr. 10	+·03	− 1·8
14	−·27	−1·4	1794			11	+·02	− 4·1
			Mar. 21	−·05	+1·6	22	−·10	+ 3·5
1791			22	−·10	+4·5	24	−·04	− 0·9
Feb. 27	−·30	+1·0	23	·00	+2·6	25	+·05	− 0·6
Mar. 1	−·04	−0·7				28	+·09	− 1·4
			1795			May 3	·00	+10·8†
1792			Mar. 20	−·01	−1·0	6	+·03	+ 1·3
Mar. 5	−·16	−1·4	22	−·01	+0·2	8	−·10	+ 3·5
8	−·06	−1·1	Apr. 3	−·03	−2·0			

* *Astronomical Papers of the American Ephemeris*, vol. vii.

† This residual should *possibly* be decreased by 13″·2, the value of one vernier division of the quadrant.

LIST OF ADDITIONS

TO THE

LIBRARY OF THE SOCIETY

1932 JUNE 30 TO 1933 JUNE 30

Serial Publications received by exchange in the regular course, and Reprints of Papers published in some of these Serials, are not included in this Annual List.

An asterisk (*) *indicates that the work is an excerpt.*

American Association of Variable Star Observers :
*Monthly Reports and Annual Report for the year 1932.
(*Association.*) 8vo. Northfield, 1932

Arakatsu (B.), Y. Ota and K. Kimura :
The Continuous Spectrum of Hydrogen associated with each of the Lines in the Balmer Series. (Memoirs of the Faculty of Science and Agriculture, Taihoku Imperial University, Vol. V, No. 1, Physics No. 1.)
(*Taihoku Imperial University.*) 4to. Formosa, 1932

Arcetri, Reale Osservatorio Astrofisico :
Pubblicazioni della R. Università degli Studi di Firenze. Osservazioni e Memorie. Indice generale dei fascicoli 1 a 50 (1896–1932).
(*Observatory.*) 8vo. Pavia, 1932

Bamberg, Remeis-Sternwarte :
Veröffentlichungen. Band 1, Heft 3.
(*Observatory.*) 4to. Bamberg, 1932
Band 1, Heft 3. E. Zinner, Die Bearbeitung von Hartwig's Beobachtungen veränderlichen Sterne.

Belgrade, Observatoire Astronomique de l'Université :
Publications, Mémoires 1.
(*Observatory.*) 8vo. Beograd, 1932

Berlin, Preussische Akademie der Wissenschaften :
Geschichte des Fixsternhimmels. Abteilung I, Band 18.
(*Purchased.*) 4to. Karlsruhe, 1932
Band 18. 17^h Rektaszension.

Bern, Schweizerische Geodätische Kommission :
Astronomisch-geodätische Arbeiten in der Schweiz. Band 19.
(*Swiss Geodetic Commission.*) 4to. Berne, 1932
Band 19. E. Hunziker, Das astronomische Nivellement im Meridian des St. Gotthard. I. Die beobachteten Lotabweichungen.

Biot (J. B.):

Recherches sur plusieurs points de l'astronomie égyptienne, appliquées aux monumens astronomiques trouvés en Egypte.
(*Solar Physics Observatory, Cambridge.*) 8vo. Paris, 1823

Bordeaux, Observatoire:

——: Carte photographique du Ciel. Zones +12°, +14°, +16°. (38 Charts.)
(*Observatory.*)

——: Catalogue photographique du Ciel. Coordonnées rectilignes. Tome 6, Zone +11° à +13°.
(*Observatory.*) 4to. Laval, 1931

——: Nouvelles observations des étoiles contenues dans le Second Catalogue de l'Observatoire de Bordeaux, publiées sous la direction de M. L. Picart. Étoiles de repère du Catalogue photographique, Zone +10° à +18°.
(*Observatory.*) 4to. Hendaye, 1932

British Association for the Advancement of Science:

Mathematical Tables. Volume II.
(*British Association Tables Committee.*) 4to. London, 1932

Vol. II. Emden Functions; being solutions of Emden's Equation together with certain associated functions.

Brussels, Institut d'Astronomie de l'Université:

——: *Publications, Série 1, No. 3.
(*University.*) 4to. Bruxelles, 1932

No. 3. P. Bourgeois et J. F. Cox, Observations astronomiques effectuées à l'Université de Bruxelles pour y constituer une station de Laplace.

——: *Publications, Série 2, No. 18–20.
(*University.*) 8vo. Bruxelles, 1932

Cambridge University, Solar Physics Observatory:

Annals, Vol. 3, Part 2.
(*Observatory.*) 4to. Cambridge, 1933

Vol. 3, Part 2. R. v. d. R. Woolley, Microphotometry of the Solar Spectrum from 4040 to 4390 A.

Canada, Geodetic Survey:

Annual Report of the Director for the Year ending March 31, 1932.
(*Director.*) 8vo. Ottawa, 1933

Canton, Observatoire de l'Université de Sun Yatsen:

Annales (1930–1931) de la commission des observateurs d'étoiles variables de la Société Astronomique de Chine.
(*Observatory.*) 4to. *s.l.*, 1932

Cape of Good Hope, Royal Observatory:

Annals, Vol. 13, Part 3.
(*Observatory.*) 4to. London, 1932

Vol. 13, Part 3. H. Spencer Jones, Discussion of observations of occultations of stars by the moon, 1672–1908, being a revision of Newcomb's "Researches on the Motion of the Moon, Part II."

Catania, Reale Osservatorio Astrofisico:
Catalogo Astrofotografico, 1900·0. Zona +46° e +55°. Vol. 3, parte 7; Vol. 4, parte 7; Vol. 5, parti 7 e 8.
(*Observatory.*) 4to. Catania, 1932–33

Coimbra University:
Revista da Faculdade de Ciências, Vol. II, No. 1.
(*Royal Society.*) 8vo. Coimbra, 1932
Vol. II, No. 1. Comemoração promovida . . . em honra de Isaac Newton.

Copenhagen, Geodetic Institute:
*Meddelelse, No. 4.
(*Geodetic Institute.*) 8vo. Leipzig, 1932

Copenhagen, Geodetic Institute, Seismological Station (Ivigtut):
Bulletin, No. 1. 1929–30.
(*Geodetic Institute.*) fol. typescript, 1930

Córdoba, Observatorio Nacional Argentino:
Resultados, Vol. 21, Entrega 5; Vols. 32, 33.
(*Observatory.*) 4to. Córdoba, 1931–32
Vol. 21, Ent. 5. Catalogo de las zonas de exploración −62° a −90°.
Vols. 32, 33. Catalogo Astrografico, zonas −30°, −31°.

Danjon (A.):
*Le déplacement apparent des étoiles autour du soleil éclipsé.
(*Author.*) 8vo. Paris, 1932

Darwin (C. G.):
The New Conceptions of Matter.
(*Purchased.*) 8vo. London, 1932

Davison (Charles):
——: *On the Eleven-year Period of Earthquake Frequency.
(*Author.*) 8vo. London, 1933

——: *On the Nineteen-year Period of Earthquake Frequency and on its Connexion with the Nutation Period of the Earth.
(*Author.*) 8vo. London, 1933

Dearborn Observatory of Northwestern University:
Annals, Vol. IV, Part 4.
(*Observatory.*) 4to. Evanston, 1933
Vol. IV, Part 4. O. J. Lee and E. G. Bryant, The effect of atmospheric refraction upon the deflection of starlight by the Sun's gravitational field.

Dietz (David):
The Story of Science.
(*Author.*) 8vo. London, 1932

Eddington (Sir Arthur):
——: *Stellar Structure.
(*Author.*) 8vo. Berlin, 1931

Eddington (Sir Arthur) :

——: *The Expanding Universe. (Presidential Address to the Physical Society of London, 1931 November 6.)
(*Author.*) 8vo. London, 1932

——: The Expanding Universe.
(*Purchased.*) 8vo. Cambridge, 1933

Edwards (Lawrence) :

The Spangled Heavens. An introduction to astronomy.
(*Publishers.*) 8vo. London, 1932

Euler (Leonhard) :

Opera Omnia; sub auspiciis Societatis Scientiarum Naturalium Helveticæ edenda curaverunt F. Rudio, A. Krazer, A. Speiser, L. G. du Pasquier. Series I. Opera Mathematica. Vol. 19.
(*Purchased.*) 4to. Turici, 1932

Vol. 19. Commentationes analyticæ ad theoriam integralium pertinentes, III. Ediderunt Alexander Liapounoff, Adolf Krazer, Georg Faber.

Fabry (Ch.) :

*La photométrie astronomique et celle des physiciens.
(*Author.*) 8vo. Paris, 1932

Feldman (W. M.) :

Rabbinical Mathematics and Astronomy. With preface by R. A. Sampson and foreword by Adolf Fraenkel.
(*Author.*) 8vo. London, 1931

Fletcher (L.) :

An introduction to the study of meteorites; with a list of the meteorites represented in the Collection [of the British Museum (Natural History)]. Eleventh edition.
(*C. O. Bartrum.*) 8vo. London, 1914

Freundlich (Erwin), Harald v. Klüber, Albert v. Brunn :

*Weitere Untersuchungen über die Bestimmung der Lichtablenkung im Schwerefeld der Sonne.
(*E. Freundlich.*) 4to. Potsdam, 1933

Gould (R. T.) :

The First Marine Timekeeper (Harrison No. 1). (Article from *The Times*, 1933 February 28.)
(*Author.*) London, 1933

Greenwich, Royal Observatory :

——: Astrographic Catalogue, 1900·0. Greenwich Section, Dec. +64° to +90°. From photographs taken and measured . . . under the direction of Sir Frank Watson Dyson, K.B.E., Sc.D., LL.D., F.R.S., Astronomer Royal. Vol. 6.
(*Observatory.*) 4to. London, 1932

Vol. 6. Proper Motions of Stars from Dec. +72° to +90°.

Greenwich, Royal Observatory:
——: Observations of colour temperatures of stars made . . . in the years 1926–1932 under the direction of Sir Frank Dyson, K.B.E., F.R.S., Astronomer Royal.
(*Observatory.*) 4to. London, 1932

Gunther (R. T.):
——: Eighth annual report for 1931 of the Committee of Management of the Lewis Evans Collection.
(*Author.*) 8vo. Oxford [1932]

——: The Astrolabes of the World: based upon a series of instruments in the Lewis Evans Collection in the Old Ashmolean Museum at Oxford, with notes on astrolabes in the collections of the British Museum, Science Museum, Sir J. Findlay, Mr. S. V. Hoffman, The Mensing Collection, and in other public and private collections.
(*Purchased.*) 2 vols. 4to. Oxford, 1931

Vol. I. The Eastern Astrolabes.
Vol. II. The Western Astrolabes.

Handbuch der Astrophysik: Herausgegeben von G. Eberhard, A. Kohlschütter, H. Ludendorff. Band V.
(*Purchased.*) 8vo. Berlin, 1932–33

Band V. Erste Hälfte. Das Sternsystem, Erster Teil, I. Bearbeitet von Fr. Becker, A. Brill, R. H. Curtiss, K. Lundmark.
—— Zweite Hälfte. Das Sternsystem, Erster Teil, II. Bearbeitet von Heber D. Curtis, B. Lindblad, K. Lundmark, H. Shapley.

Heath (Sir Thomas L.):
Greek Astronomy.
(*Publishers.*) 8vo. London, 1932

Helsingfors, Baltic Geodetic Commission:
Special Publication, No. 2.
(*Commission.*) 8vo. Helsinki, 1932

No. 2. Bestimmung der Längenunterschiede der Landeszentralen im Jahre 1929. I. Das Beobachtungsmateriel.

Helsingfors, Observatoire:
Catalogue photographique du Ciel. Tome VIII, fasc. 2.
(*Observatory.*) 4to. Helsingfors, 1932

Tome VIII, fasc. 2. Clichés de $22^h\ 30^m$ à $24^h\ 0^m$.

Iba (Yasuaki):
——: Fragmentary Notes on Astronomy in Japan.
(*Author.*) 8vo. *s.l.*, 1932

——: Appendix to Fragmentary Notes on Astronomy in Japan.
(*Author.*) 8vo. *s.l.*, 1932

India, Survey Department:
Professional Papers. No. 27.
(*Surveyor General of India.*) 8vo. Dehra Dun, 1932

No. 27. E. A. Glennie, Gravity Anomalies and the Structure of the Earth's Crust.

International Astronomical Union:

——: Bulletin for Character Figures of Solar Phenomena: Published by the Eidgenossische Sternwarte in Zürich. Nos. 17–21.
(*I.A.U.*) 4to. Zürich, 1932–33

——: Transactions. Vol. 4, 1932.
(*I.A.U.*) 4to. Cambridge, 1933

International Geodetic and Geophysical Union:

Association d'océanographie physique. *Publication Scientifique. Nos. 1, 2.
(*I.G. and G. Union.*) 8vo. Helsingfors, 1931–32

No. 1. S. F. Grace, I. Historical Review of Dynamical Explanations of Tides in Non-elongated Enclosed Seas and Lakes.
II. Historical Review of Dynamical Explanations of the Tides of the Mediterranean, the Baltic Sea, the Gulf of Mexico and the Arctic Ocean.
No. 2. J. Proudman, Tidal Bibliography.

Jackson (J.):

——: The Sun at Greenwich Mean Noon, 1775–1798
(*Author.*) fol. typescript, 1933

——: The Sun at Oxford Apparent Noon, 1775–1798.
(*Author.*) fol. typescript, 1933

Jeans (Sir James):

The New Background of Science.
(*Author.*) 8vo. Cambridge, 1933

Klüber (H. v.):

——: *Über eine Horizontalkamera mit Präzisions-Coelostaten für astronomische Beobachtungen.
(*Author.*) 8vo. Berlin, 1932

——: *Photometrische Messungen in der atmosphärischen Sauerstoffbande λ 7600 (A-Gruppe) im Sonnenspektrum.
(*Author.*) 8vo. Berlin, 1933

Krat (W.):

——: *Über die Ableitung des Randverdunkelungsgesetzes der Algolsterne.
(*Author.*) 8vo. Berlin, 1932

——: *Über die Ableitung des Randverdunkelungsgesetzes des β-Lyrae-Sterne.
(*Author.*) 8vo. Berlin, 1933

Leadbetter (Charles):

A Treatise of Eclipses of the Sun and Moon for Thirty-five Years, commencing Anno 1715, ending 1749. The Second Edition, with Additions.
(*F. J. M. Stratton.*) 8vo. London, 1731

Leningrad, Astronomical Institute:

Catalogue and Ephemerides of Short-period Cepheids for 1932.
(*Institute.*) 8vo. Leningrad, 1932

Liège Université, Institut d'Astronomie et de Géodésie :
Physique du Globe, No. 1.
(*Institute.*) 4to. Bruxelles, 1931

London, Meteorological Office :
Geophysical Memoirs, No. 58.
(*Director.*) 4to. London, 1933
No. 58. F. J. Scrase, The Air-Earth Current at Kew Observatory. Some Results obtained with a New Automatic Recorder.

London, Nautical Almanac Office :
Planetary Co-ordinates for the years 1800–1940, referred to the equinox of 1950·0. Prepared by H.M. Nautical Almanac Office.
(*L. J. Comrie.*) 4to. London, 1933

London, Optical Society :
Transactions. Vol. 33, Nos. 3–5.
(*Optical Society.*) 8vo. Cambridge, 1932

Continued as **London, The Physical Society :**
Proceedings. Vol. 45, Parts 1–3.
(*Physical Society.*) 8vo. Cambridge, 1933

London, Royal Society :
Index to the Proceedings of the Royal Society of London (1905–1930), and to the Philosophical Transactions of the Royal Society of London (1901–1930).
(*Purchased.*) 8vo. London, 1932

London, Science Museum :
Report of the Advisory Council of the Science Museum for the year 1931.
(*Director.*) 8vo. London, 1932

Lund, Astronomiska Observatorium :
Annals. No. 3.
(*Observatory.*) 4to. Malmo and Lund, 1932
No. 3. Tables for the conversion of equatorial co-ordinates into Galactic co-ordinates based on the Galactic Pole R.A. $12^h\,40^m$ Dec. $+28°$ (1900·0). Computed under the direction of and provided with an introduction by John Ohlsson.

Lwów University, Astronomical Institute :
Contributions, Nr. 1.
(*Institute.*) 8vo. Lwów, 1933

Lyons, Association Française d'Observateurs d'Etoiles Variables :
Bulletin, Tome 1, Fascs. 2–6; Tome 2, Fasc. 1.
(*Purchased.*) 4to. Lyon, 1932–33

Lyons, Observatoire :
*Publications. Série I, Astronomie. Tome 1, Fasc. 1, 2.
(*Observatory.*) 4to. *v.l.*, [1932]

Macelwane (J. B.) and F. W. Sohon:
Introduction to Theoretical Seismology. Part II.
(*Purchased.*) 8vo. New York, 1932
Part II. F. W. Sohon, S.J., Seismometry.

Macpherson (Hector):
Makers of Astronomy.
(*Author.*) 8vo. Oxford, 1933

Madrid, Observatorio:
Boletín Astronómico. Vol. 1, Núm. 7–15.
(*Observatory.*) 8vo. Madrid, 1932-33

Milan, Reale Osservatorio di Brera:
Pubblicazioni. No. 62.
(*Observatory.*) 4to. Milano, 1932
No. 62. P. Vocca e F. Zagar, Differenza di longitudine Milano-Zurigo determinata nell' anno 1929.

Milne (E. A.):
*World-structure and the Expansion of the Universe.
(*Author.*) 8vo. Berlin, 1933

Moffitt (G. W.):
*Frequency requirements and the control of frequency for synchronous motor operation of astronomical telescopes.
(*Yerkes Observatory.*) 8vo. Menasha, 1932

Mohr (Jenka):
A Brief Account of the Harvard Observatory.
(*Harvard Observatory.*) 8vo. Cambridge (Mass.), 1932

Müller (Rolf):
——: *Die astronomische Bedeutung des Mecklenburgischen "Steintanzes" bei Bützow.
(*Author.*) 8vo. *s.l.*, 1931

——: *Über die Dunkelnebel bei ξ Ophiuchi. Abzählung in Dunkelfeldern der südlichen Milchstrasse (3).
(*Author.*) 8vo. Berlin, 1932

——: *Frühere Minima von ζ Aurigae.
(*Author.*) 4to. Kiel, 1932

Nilsson (Gerhard):
——: Das Doppelmolekül.
(*Author.*) 4to. [Kopenhagen], 1932

——: Die Temperaturen im Weltraume.
(*Author.*) 4to. Stockholm, 1932

——: Der Bau des Universums.
(*Author.*) 4to. Stockholm, 1933

Ottawa, Dominion Observatory:

——: Dominion Observatory Reprints, Nos. 11, 15, 17, 18.
(*Observatory.*) 8vo. *v.l.*, 1932–33

No. 11. Ernest A. Hodgson, The *P*-curve and the *S*-curve resulting from a study of the Tango Earthquake, Japan, March 7, 1927.
No. 15. Ernest A. Hodgson, The Foundations of Earth-structure Theory.
No. 17. Ernest A. Hodgson, Two Probability Methods for the Determination of Earthquake Epicentres.
No. 18. Ernest A. Hodgson, Epicentral Time and Surface Structure determined for the Tango Earthquake, Japan, March 7, 1927.

——: Dominion Observatory Pamphlets, M 4, 6.
(*The Minister of the Interior.*) 4to. typescript, 1932

Paris, Comité National Français d'Astronomie:

Congrès national d'Astronomie des 21–22–23 Juillet 1932.
(*Committee.*) 8vo. Laval, *s.d.*

Paris, Institut de France: Académie des Sciences:

——: Annuaire pour 1933.
(*Academy.*) 16mo. Paris, 1932

——: Mémoires. Deuxieme Série. Tome 60.
(*Academy.*) 4to. Paris, 1931

——: Bulletin bibliographique des publications périodiques reçues pendant l'année 1932.
(*Academy.*) 4to. Paris, 1933

——: Les membres et les correspondants de l'Académie Royale des Sciences, 1666–1793.
(*Academy.*) 16mo. Paris, 1931

Paris, Observatoire:

Carte photographie du Ciel. Zones +18°, +20°. (25 Charts.)
(*Observatory.*)

Pasadena, California Institute of Technology:

Bulletin, Vol. 42, No. 138.
(*Institute.*) 8vo. Pasadena, 1933

Vol. 42, No. 138. Albert Einstein, Henry M. Robinson, William B. Munro, Symposium on America and the world situation.

Peiping [Pekin], National Tsing Hua University:

The Science Reports. Series A: Mathematical and Physical Sciences. Vol. 1, Nos. 5, 6; Vol. 2, No. 1.
(*University.*) 8vo. Peiping, 1932–33

Perepelkin (E. J.):

——: Über die Rotationsgeschwindigkeit der verschiedenen Schichten der Sonne
(*Author.*) 8vo. Berlin, 1933

——: Über die Struktur der Sonnenchromosphäre.
(*Author.*) 8vo. Berlin, 1933

Perkins Observatory, Ohio Wesleyan University:
*Miscellaneous Scientific Papers. Reprint Nos. 12, 13, 15.
(*Observatory.*)

No. 12. H. T. Stetson, Further Studies on the Lunar Correlations with small changes in the Variation of Latitude.
No. 13. H. T. Stetson, Progress in the Studies of Cosmic Correlations with Radio Reception at the Perkins Observatory.
No. 15. N. T. Bobrovnikoff, On the Phenomena of Halos in Comets.

Pickering (W. H.):
*Pluto, the Ninth Planet.
(*Author.*) 8vo. Cambridge (Mass.), 1932

Pinto (Olga):
Termini d'uso nelle bibliografie dei periodici. Saggio compilato in 38 lingue. (Primo congresso mondiale delle bibliotheche e di bibliografia.)
(*Purchased.*) 8vo. Roma, 1929

Potsdam, Preussisches Geodätisches Institut:
Ergebnisse des Internationalen Breitendienstes von 1912·0 bis 1922·7 von B. Wanach und H. Mahnkopf.
(*Institute.*) 4to. Potsdam, 1932

Prandtl (W.):
Die Bibliothek des Tycho Brahe.
(*Purchased.*) 8vo. Wien, 1933

Princeton University Observatory:
A Multiplet Table of Astrophysical Interest. By Charlotte E. Moore.
(*Observatory.*) 4to. Princeton, N.J., 1933

Pulkovo, Observatoire:
Circular, Nos. 1–6.
(*Observatory.*) 4to. Leningrad, 1932–33

Rome, Primo Congresso Mondiale delle Bibliotheche e di Bibliografia:
See Pinto (Olga).

Rome, Reale Accademia d'Italia:
Memorie. Classe di Scienze fisiche, matematiche e naturali. Ingegneria. Vol. 3, No. 2.
(*Academy.*) 8vo. Roma, 1932

Rome, Reale Accademia d'Italia, Fondazione Alessandro Volta:
——: Atti dei Convegni, No. 1.
(*Academy.*) 8vo. Roma, 1932

No. 1. Convegno di Fisica Nucleare, Ottobre 1931 IX.

——: Viaggi di Studio ed Esplorazioni I.
(*Academy.*) 8vo. Roma, 1932

Russell (Henry Norris):
The Composition of the Stars: being the Halley Lecture delivered on June 1, 1933.
(*Author.*) 8vo. Oxford, 1933

Scott (John) :
*The Adjustments of an Equatorial Mounting for Astronomical Telescopes.
(*Author.*) 8vo. [York], 1932

Sengupta (P. C.) :
——: *Aryabhata's Lost Work.
(*Author.*) 8vo. Calcutta, 1930

——: *Brahmagupta on Interpolation.
(*Author.*) 8vo. Calcutta, 1931

——: *Infinitesimal Calculus in Indian Mathematics—its Origin and Development.
(*Author.*) 8vo. Calcutta, 1931

——: *Hindu Luni-Solar Astronomy.
(*Author.*) 8vo. Calcutta, 1932

Silberrad (C. A.) :
*List of Indian Meteorites.
(*L. J. Spencer.*) 8vo. London, 1932

Simpson (G. C.) :
Physics in Meteorology. Physics in Industry Lecture No. 18.
(*Institute.*) 8vo. London, 1932

Sitter (W. de) :
——: *On the expanding universe.
(*Author.*) 8vo. Amsterdam, 1932

——: Kosmos. A Course of Six Lectures on the Development of our Insight into the Structure of the Universe, delivered for the Lowell Institute in Boston, in November 1931.
(*Purchased.*) 8vo. Cambridge (Mass.), 1932

Slipher (V. M.) :
*The Lowell Observatory.
(*Author.*) 8vo. San Francisco, 1927

Spencer (L. J.) :
——: *Meteorite Craters as Topographical Features on the Earth's Surface.
(*Author.*) 8vo. London, 1933

——: *Origin of Tectites.
(*Author.*) 8vo. London, 1933

——: *A new meteoric iron found near Kyancutta, South Australia. With a chemical analysis by M. H. Hey.
(*Author.*) 8vo. London, 1933

Strasbourg, Observatoire :
Annales. Tome II, fasc. 2.
(*Observatory.*) 4to. Orléans, 1932

Strömberg (Gustaf) :
*Space Structure and Motion.
(*Author.*) 4to. *s.l.*, 1932

Tachkent, Astronomical Observatory :
Astronomical Circular, Nos. 1–15.
(*Observatory.*) 8vo. Tachkent, 1932–33

Tachkent, Universitas Asiæ Mediæ :
Acta. Series V*b*. Astronomia, fasc. 5
(*University.*) 8vo. Tachkent, 1932

Tams (E.) :
*Einige Korrelationen zwischen seismischer Bodenunruhe im Hamburg und der Brandung in West- und Nordeuropa.
(*Hamburg, Erdbeben und Physikalisches Staatinstitut.*)
8vo. Braunschweig, 1933

Taylor (H. E.) :
Wonders of the Universe. A simple account of the elements of astronomy.
(*Author.*) 8vo. London, 1933

Tiflis, Stalin Bau-Institut der S.S.R. Georgien :
Berichte. Heft 1.
(*Institute.*) 8vo. Tiflis, 1932

Toronto, Royal Astronomical Society of Canada :
General Index to the Publications of the Astronomical and Physical Society of Toronto 1890–1899, Toronto Astronomical Society 1900–1901, Royal Astronomical Society of Canada 1902–1931; being the Transactions 1890–1905, Journal 1907–1931 (Vols. 1–25 inclusive). Compiled by W. E. Harper, Assistant Director, Dominion Astrophysical Observatory, Victoria, Canada.
(*Purchased.*) 8vo. Toronto, 1932

Toulouse, Observatoire Astronomique :
Carte photographique du Ciel. Zones +5°, +7°, +9°. (7 Charts.)
(*Observatory.*)

Uccle, Observatoire Royal de Belgique :
——: Bulletin Astronomique, 2me Année, 1932. Vol. 1, Nos. 4–8.
(*Observatory.*) 4to. Gembloux, 1932

——: Carte photographique du Ciel. Zones +33°, +35°, +37°, +39°. (20 Charts.)
(*Observatory.*)

Ungerer (Alfred) :
Les horloges astronomiques et monumentales les plus remarquables de l'antiquité jusqu'à nos jours. Avec une préface par E. Esclangon. En appendice: Table chronologique, table onomastique, table bibliographique.
(*Purchased.*) 4to. Strasbourg, 1931

Van de Kamp (P.) :
*On the absorption of light in the Galactic System.
(*Leander McCormick Observatory*.) 4to. Albany, 1932

Volta (Luigi) :
Le Stelle. Discorso letto il 5 Novembre 1930—IX—nell' Aula Magna della R. Università di Torino per la solenne inaugurazione dell' Anno Accademico.
(*Pino Torinese Observatory*.) 8vo. Torino, 1931

Volterra (Vito) :
Le calcul des variations, son évolution et ses progrès, son rôle dans la physique mathématique.
(*Masaryk University*.) 8vo. Brno, 1932

Wainwright (G. A.) :
*A pair of constellations. (With map by E. Williamson.)
(*G. A. Wainwright and Miss E. Williamson*.) 4to. [London], [1932]

Warsaw, Institut d'Astronomie Pratique de l'Ecole Polytechnique :
Publications, Nos. 8–12.
(*Institute*.) 8vo. Cracovie, Warszawa, 1932–33

Washington, Carnegie Institution :
——: *Annual Report of the Director of the Department of Terrestrial Magnetism for the year 1931–32.
(*Carnegie Institution of Washington*.) 8vo. Washington, 1932

——: List of publications of the Department of Terrestrial Magnetism, 1932.
(*Carnegie Institution of Washington*.) 8vo. Washington, 1932

——: Report of the President of the Carnegie Institution of Washington for the year ending October 31, 1932.
(*Carnegie Institution of Washington*.) 8vo. Washington, [1932]

West (Carlos) :
Movimento central de los astros.
(*Author*.) 8vo. Buenos Aires, 1932

Whyte (Charles) :
Stellar Wonders.
(*Author*.) 8vo. London, 1933

Wilkens (A.) :
*Über das Problem der mehrfachen Kommensurabilitäten im Sonnensystem.
(*Author*.) 8vo. München, 1933

Williams (W. Ewart) :
Applications of Interferometry. With a preface by O. W. Richardson, F.R.S.
(*Purchased*.) 8vo. London, 1930

Zinner (E.) :

——: *Merkwürdige mittelalterliche Sonnenuhren.
(*Author*.) 8vo. Berlin, 1932

——: *Die astrologische Treffsicherheit.
(*Author*.) 4to. Berlin, 1932

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MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 93 No. 1 1932 November

Published and Sold by the

ROYAL ASTRONOMICAL SOCIETY
BURLINGTON HOUSE
LONDON, W. 1

Price Four Shillings

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Printed in Edinburgh by Neill & Co., Ltd., 1933 January